A TREATISE OF FLUXIONS.

In Two BOOKS.

BY

COLIN MACLAURIN, A. M.

Professor of Mathematics in the University of Edinburgh, *and Fellow of the Royal Society.*

VOLUME I.

EDINBURGH:
Printed by T. W. and T. RUDDIMANS.
MDCCXLII.

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MVSEVM BRITAN NICVM

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To His GRACE

The Duke of ARGYLE

AND

GREENWICH.

MY LORD,

IT is not the ſhining Figure your GRACE has made in the higheſt Employments, neither is it the Luſtre which your quitting them has added to your Character; it is not the Favour of Princes which you have often enjoy'd, nor the loud and univerſal Applauſes of the People which you at preſent poſſeſs; but it

is the ſteady Virtue which has conducted you throughout, that determines me to offer this Work to your GRACE. And, having been honoured with the Countenance and Favour of ſo great a Man, I embrace the Opportunity of expreſſing my Gratitude at a Time when I can have no other Motive than that I am, with Truth, and the utmoſt Reſpect,

MY LORD,

Your Grace's

much obliged,

moſt faithful,

and moſt humble Servant,

COLIN MACLAURIN.

PREFACE.

A *Letter published in the Year* 1734, *under the Title of* the Analyst, *first gave Occasion to the ensuing Treatise, and several Reasons concurred to induce me to write on this Subject at so great length. The Author of that Piece had represented the Method of Fluxions as founded on false Reasoning, and full of Mysteries. His Objections seemed to have been occasioned in a great measure by the concise Manner in which the Elements of this Method have been usually described; and their having been so much misunderstood by a Person of his Abilities appeared to me a sufficient Proof that a fuller Account of the Grounds of them was requisite.*

Though there can be no Comparison made betwixt the Extent or Usefulness of the ancient and modern Discoveries in Geometry, yet it seems to be generally allowed that the Ancients took greater Care, and were more successful in preserving the Character of its Evidence entire. This determined me, immediately after that Piece came to my Hands, (and before I knew any thing of what was intended by others in answer to it,) to attempt to deduce those Elements

ments after the Manner of the Antients, from a few unexceptionable Principles, by Demonſtrations of the ſtricteſt Form. In my firſt Eſſay of this Kind, I contented myſelf with demonſtrating the principal Caſes of the Propoſitions of the four firſt Chapters of the firſt Book, and of the firſt Chapter of the ſecond Book of the following Treatiſe, nearly in the ſame Form in which they now appear. But when it was communicated to ſome Gentlemen, they expreſſed a Deſire that the ſame Method of Demonſtration ſhould be extended to other Branches of this Theory, and that I ſhould enlarge the Plan. While I proceeded in this Work, I perceived that ſome Rules were defective or inaccurate; that the Reſolution of ſeveral Problems which had been deduced in a myſterious manner, by ſecond and third Fluxions, could be compleated with greater Evidence, and leſs Danger of Error, by firſt Fluxions only; and that other Problems had been reſolved by Approximations, when an accurate Solution could be obtained with the ſame or greater Facility. Theſe with other Obſervations concerning this Method, and its Application, led me on gradually to compoſe a Treatiſe of a much greater Extent than I intended, or would have engaged in, if I had been aware of it when I began this Work, becauſe my Attendance in the Univerſity could allow me to beſtow but a ſmall part of my Time in carrying it on. And as this has been the Occaſion of my Delay in publiſhing it, ſo I hope it will ſerve for an Apology, if ſome Miſtakes have eſcaped me in treating of ſuch a Variety of Subjects, in a manner different from that in which they have been uſually explained.

In

In the mean Time the Defence of the Method of Fluxions, and of the great Inventor, was not neglected. Beſides an Anſwer to the Analyſt *that appeared very early under the Name of* Philalethes Cantabrigienſis, (*for the Author had concealed his real Name, as the* Analyſt *whom he oppoſed had done,*) *a ſecond by the ſame Hand in Defence of the firſt, a Diſcourſe by Mr.* Robins, *a Treatiſe of Sir* Iſaac Newton's *with a Commentary by Mr.* Colſon, *and ſeveral other Pieces were publiſhed on this Subject. After I ſaw that ſo much had been written upon it to ſo good Purpoſe, I was the rather induced to delay the Publication of this Treatiſe, till I could finiſh my Deſign. I accommodated my Definition of the Variation of Curvature in Chap.* xi. *to Sir* Iſaac Newton's, *to prevent Miſtakes, as I have obſerved in Article* 386. *but made no material Alteration in any thing elſe. The greateſt Part of the firſt Book was printed in* 1737: *But it could not have been ſo uſeful to the Reader without the ſecond; and I would recommend to him, (if he is not already acquainted with this Method,) to peruſe the two firſt Chapters of the ſecond Book, before the five laſt of the firſt; there being a few Paſſages in theſe which I could not well avoid, that will be better underſtood by one who has ſome Knowledge of the principal Rules of the Method of Computation delivered in the ſecond Book.*

In explaining the Notion of a Fluxion, I have followed Sir Iſaac Newton *in the firſt Book, imagining that there can be no Difficulty in conceiving Velocity wherever there is Motion; nor do I think that I have de-*

departed from his Senſe in the ſecond Book; and in both I have endeavoured to avoid ſeveral Expreſſions, which, though convenient, might be liable to Exceptions, and, perhaps, occaſion Diſputes. I have always repreſented Fluxions of all Orders by finite Quantities, the Suppoſition of an infinitely little Magnitude being too bold a Poſtulatum *for ſuch a Science as Geometry. But, becauſe the Method of Infiniteſimals is much in uſe, and is valued for its Conciſeneſs, I thought it was requiſite to account explicitely for the Truth, and perfect Accuracy, of the Concluſions that are derived from it; the rather, that it does not ſeem to be a very proper Reaſon that is aſſigned by Authors, when they determine what is called the* Difference (*but more accurately the Fluxion*) *of a Quantity, and tell us, That they reject certain Parts of the Element, becauſe they become infinitely leſs than the other Parts; not only, becauſe a Proof of this Nature may leave ſome Doubt as to the Accuracy of the Concluſion; but becauſe it may be demonſtrated that thoſe Parts ought to be neglected by them at any Rate, or that it would be an Error to retain them. If an Accountant, that pretends to a ſcrupulous Exactneſs, ſhould tell us that he had neglected certain Articles, becauſe he found them to be of ſmall Importance; and it ſhould appear that they ought not to have been taken into Conſideration by him on that Occaſion, but belong to a different Account, we ſhould approve his Concluſions as accurate, but not his Reaſon. This Method, however, may be conſidered as an eaſy and ready Way of diſtinguiſhing what Parts of an Element are to be rejected, and which are to be retained, in determining the preciſe Fluxion of a*

Quan-

Quantity, or the Rate according to which it increaſes or decreaſes.

After I found that this Treatiſe could not be conveniently contained in one Volume, I was obliged to reprint two Leaves (Pages 411. &c.) *that it might be divided into two. I have reprinted likewiſe the firſt Sheet, chiefly on account of ſeveral Errors of the Preſs that had got into it, and one other Leaf (p.*244.) *for the ſake of a Paſſage, the Omiſſion of which poſſibly would have been miſintepreted. There are ſome Demonſtrations in the firſt Chapter of the firſt Book that might have been abridg'd, and ſome, perhaps, will appear unneceſſary. I have mentioned the Reaſons that induced me to inſiſt ſo fully on thoſe elementary Parts in Articles* 69, 104. 494. *and* 697.

Several Treatiſes have appeared while this was in the Preſs, wherein ſome of the ſame Problems have been conſidered, though generally in a different manner. I have had Occaſion to mention moſt of them, in the laſt Chapter of the ſecond Book; but had not there an Opportunity to take notice, that the Problem in 480. *has been conſidered by Mr.* Euler *in his Mechanics. In moſt of the Inſtances wherein my Concluſions did not agree with thoſe given by other Authors, I have not mentioned their Names.*

If upon the whole, the Evidence of this Method be repreſented to the Satisfaction of the Reader, ſome of the abſtruſe Parts illuſtrated, or any Improvements of this uſeful Art be propoſed, I ſhall be under no great Concern, though Exceptions may

be made to ſome Modes of Expreſſion, or to ſuch Paſſages of this Treatiſe as are not eſſential to the principal Deſign.

INTRO-

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INTRODUCTION.

GEOMETRY is valued for its extenſive uſefulneſs, but has been moſt admired for its evidence; mathematical demonſtration being ſuch as has been always ſuppoſed to put an end to diſpute, leaving no place for doubt or cavil. It acquired this character by the great care of the old writers, who admitted no principles but a few ſelf-evident truths, and no demonſtrations but ſuch as were accurately deduced from them. The ſcience being now vaſtly enlarged, and applied with ſucceſs to philoſophy and the arts, it is of greater importance than ever that its evidence be preſerved perfect. But it has been objected on ſeveral occaſions, that the modern improvements have been eſtabliſhed for the moſt part upon new and exceptionable maxims, of too abſtruſe a nature to deſerve a place amongſt the plain principles of the ancient geometry: And ſome have proceeded ſo far as to impute falſe reaſoning to thoſe authors who have contributed moſt to the late diſcoveries, and have at the ſame time been moſt cautious in their manner of deſcribing them.

In the method of indiviſibles, lines were conceived to be made up of points, ſurfaces of lines, and ſolids of ſurfaces; and ſuch ſuppoſitions have been employed by ſeveral ingenious men for proving the old theorems, and diſcovering new ones, in a brief and eaſy manner. But as this doctrine was inconſiſtent with the ſtrict principles of geometry, ſo it ſoon appeared that there was ſome danger of its leading them into falſe concluſions: Therefore others, in the place of indiviſible, ſubſtituted infinitely ſmall diviſible elements, of which they ſuppoſed all magnitudes to be formed; and thus endeavoured to retain, and improve, the advantages that were derived from the former method for the advancement of geometry. After theſe came to be reliſhed, an infinite ſcale of infinites and infiniteſimals, (aſcending and deſcending always by infinite ſteps,) was imagined, and propoſed to be received into geometry, as of the greateſt uſe for penetrating into its abſtruſe parts. Some have argued for quan-

tities more than infinite; and others for a kind of quantities that are ſaid to be neither finite nor infinite, but of an intermediate and indeterminate nature.

This way of conſidering what is called the ſublime part of geometry has ſo far prevailed, that it is generally known by no leſs a title than the *Science*, *Arithmetic*, or *Geometry of infinites*. Theſe terms imply ſomething lofty, but myſterious; the contemplation of which may be ſuſpected to amaze and perplex, rather than ſatisfy or enlighten the underſtanding, in the proſecution of this ſcience; and while it ſeems greatly to elevate geometry, may poſſibly leſſen its true and real excellency, which chiefly conſiſts in its perſpicuity and perfect evidence: For we may be apt to reſt in an obſcure and imperfect knowledge of ſo abſtruſe a doctrine, as better ſuited to its nature, inſtead of ſeeking for that clear and full view we ought to have of geometrical truth; and to this we may aſcribe the inclination which has appeared of late for introducing myſteries into a ſcience wherein there ought to be none.

There were ſome, however, who diſliked the making much uſe of infinites and infiniteſimals in geometry. Of this number was Sir ISAAC NEWTON (whoſe caution was almoſt as diſtinguiſhing a part of his character as his invention) eſpecially after he ſaw that this liberty was growing to ſo great a height. In demonſtrating the grounds of the method of fluxions * he avoided them, eſtabliſhing it in a way more agreeable to the ſtrictneſs of geometry. He conſidered magnitudes as generated by a flux or motion, and ſhowed how the velocities of the generating motions were to be compared together. There was nothing in this doctrine but what ſeemed to be natural and agreeable to the ancient geometry. But what he has given us on this ſubject being very ſhort, his conciſeneſs may be ſuppoſed to have given ſome occaſion to the objections which have been raiſed againſt his method.

When the certainty of any part of geometry is brought into queſtion, the moſt effectual way to ſet the truth in a full light, and

* De quadrat. curvarum.

and to prevent diſputes, is to deduce it from axioms or firſt principles of unexceptionable evidence, by demonſtrations of the ſtricteſt kind, after the manner of the antient geometricians. This is our deſign in the following treatiſe; wherein we do not propoſe to alter Sir ISAAC NEWTON's notion of a fluxion, but to explain and demonſtrate his method, by deducing it at length from a few ſelf-evident truths, in that ſtrict manner: and, in treating of it, to abſtract from all principles and poſtulates that may require the imagining any other quantities but ſuch as may be eaſily conceived to have a real exiſtence. We ſhall not conſider any part of ſpace or time as indiviſible, or infinitely little; but we ſhall conſider a point as a term or limit of a line, and a moment as a term or limit of time: Nor ſhall we reſolve curve lines, or curvilineal ſpaces, into rectilineal elements of any kind. In delivering the principles of this method, we apprehend it is better to avoid ſuch ſuppoſitions: but after theſe are demonſtrated, ſhort and conciſe ways of ſpeaking, though leſs accurate, may be permitted, when there is no hazard of our introducing any uncertainty or obſcurity into the ſcience from the uſe of them, or of involving it in diſputes. The method of demonſtration, which was invented by the author of fluxions, is accurate and elegant; but we propoſe to begin with one that is ſomewhat different; which, being leſs removed from that of the ancients, may make the tranſition to his method more eaſy to beginners, (for whom chiefly this treatiſe is intended,) and may obviate ſome objections that have been made to it.

But, before we proceed, it may be of uſe to conſider the ſteps by which the ancients were able, in ſeveral inſtances, from the menſuration of right-lin'd figures, to judge of ſuch as were bounded by curve lines: for as they did not allow themſelves to reſolve curvilineal figures into rectilineal elements, it is worth while to examine by what art they could make a tranſition from the one to the other: And as they were at great pains to finiſh their demonſtrations in the moſt perfect manner, ſo by following their example, as much as poſſible in demonſtrating a method ſo much more general than theirs, we may beſt guard againſt exceptions and cavils, and vary leſs from the old foundations of geometry.

They found, that ſimilar triangles are to each other in the duplicate ratio of their homologous ſides; and, by reſolving ſimilar polygons into ſimilar triangles, the ſame propoſition was extended to theſe polygons alſo. But when they came to compare curvilineal figures, that cannot be reſolved into rectilineal parts, this method failed. Circles are the only curvilineal plane figures conſidered in the elements of geometry. If they could have allowed themſelves to have conſidered theſe as ſimilar polygons of an infinite number of ſides, (as ſome have done who pretend to abridge their demonſtrations,) after proving that any ſimilar polygons inſcribed in circles are in the duplicate ratio of the diameters, they would have immediately extended this to the circles themſelves; and would have conſidered the ſecond propoſition of the twelfth book of the Elements as an eaſy corollary from the firſt. But there is ground to think that they would not have admitted a demonſtration of this kind. It was a fundamental principle with them, that the difference of any two unequal quantities, by which the greater exceeds the leſſer, may be added to itſelf till it ſhall exceed any propoſed finite quantity of the ſame kind: and that they founded their propoſitions concerning curvilineal figures upon this principle in a particular manner, is evident from the demonſtrations, and from the expreſs declaration of ARCHIMEDES, who acknowledges it to be the foundation upon which he eſtabliſhed his own diſcoveries *, and cites it as aſſumed by the ancients in demonſtrating all their propoſitions of this kind. But this principle ſeems to be inconſiſtent with the admitting of an infinitely little quantity or difference, which, added to itſelf any number of times, is never ſuppoſed to become equal to any finite quantity whatſoever.

They proceeded therefore in another manner, leſs direct indeed, but perfectly evident. They found that the inſcribed ſimilar polygons, by increaſing the number of their ſides continually approached to the areas of the circles; ſo that the decreaſing differences betwixt each circle and its inſcribed polygon, by ſtill further and further diviſions of the circular arches which the

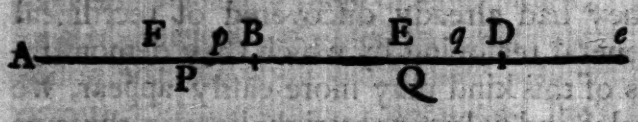
* Δείκνυται γὰρ ὅτι πᾶν τμῆμα περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογωνίου κώνου τομῆς ἐπίτριτον ἐστὶ τοῦ τριγώνου, &c. *Archimed. de quadr. parab. ad Doſith.*

the ſides of the polygons ſubtend, could become leſs than any quantity that can be aſſigned: and that all this while the ſimilar polygons obſerved the ſame conſtant invariable proportion to each other, *viz.* that of the ſquares of the diameters of the circles. Upon this they founded a demonſtration, that the proportion of the circles themſelves could be no other than that ſame invariable ratio of the ſimilar inſcribed polygons: of which we ſhall give a brief abſtract, that it may appear in what manner they were able, in this inſtance, and ſome others of the ſame nature, to form a demonſtration of the proportions of curvilineal figures, from what they had already diſcovered of rectilineal ones. And that the general reaſoning by which they demonſtrated all their theorems of this kind may more eaſily appear, we ſhall repreſent the circles and polygons by right lines, in the ſame manner as all magnitudes are expreſſed in the fifth book of the Elements.

Suppoſe the right lines AB and AD to repreſent the two areas of the circles that are compared together; and let AP, AQ repreſent any two ſimilar polygons inſcribed in theſe circles. By further continual ſubdiviſions of the circular arches which the ſides of the polygons ſubtend, the areas of the polygons increaſe, and may approach to the circles AB and AD ſo as to differ from them by leſs than any aſſignable meaſure; the triangle which is ſubducted from each ſegment at every new ſubdiviſion being always greater than the half of the ſegment. The polygons inſcribed in the two circles, as they increaſe, are ever in the ſame conſtant proportion to each other: and this invariable ratio of theſe polygons muſt alſo be the ratio of the circles themſelves. For, if it is not, let the ratio of the polygons AP and AQ to each other be, in the firſt place, the ſame as the ratio of the circle AB to any magnitude AE leſs than the circle AD; ſuppoſe the ſubdiviſions of the arches of the circle AD to be continued till the difference betwixt the circle and inſcribed polygon become leſs than ED, ſo that the polygon may be repreſented by A*q*, greater than AE; and let A*p* repreſent a polygon inſcribed in the circle AB, ſimilar to the po-

A P *p* B Q E *q* D

polygon Aq. Then, ſince AP is to AQ as AB is to AE by the ſuppoſition, and the polygon Ap is to the ſimilar polygon Aq as AP is to AQ; it follows, that AB is to AE as Ap is to Aq; and that the circle AB being greater than Ap, a polygon inſcribed in it, AE muſt be greater than Aq. But Aq is ſuppoſed to be greater than AE; and theſe being repugnant, it follows, that the polygon AP is not to the polygon AQ as the circle AB is to any magnitude (as AE) leſs than the circle AD. For the ſame reaſon AQ is not to AP as AD is to any magnitude (as AF) leſs than AB. From which it follows that we cannot ſuppoſe AP to be to AQ as AB is to any magnitude Ae greater than AD; becauſe if we take AF to AB as AD is to Ae, AF will be leſs than AB, and AP will be to AQ as AF leſs than AB to AD; againſt what has been demonſtrated. It follows therefore that AP is not to AQ as AB is to any magnitude greater or leſs than AD; but that the ratio of the circles AB and AD to each other, muſt be the ſame as the invariable ratio of the ſimilar polygons AP and AQ inſcribed in them, which is the duplicate of the ratio of their diameters.



In the ſame manner the ancients have demonſtrated, that pyramids of the ſame height are to each other as their baſes, that ſpheres are as the cubes of their diameters, and that a cone is the third part of a cylinder on the ſame baſe and of the ſame height. In general, it appears from this demonſtration, that when two variable quantities, AP and AQ, which always are in an invariable ratio to each other, approach at the ſame time to two determined quantities, AB and AD, ſo that they may differ leſs from them than by any aſſignable meaſure, the ratio of theſe limits AB and AD muſt be the ſame as the invariable ratio of the quantities AP and AQ: and this may be conſidered as the moſt ſimple and fundamental propoſition in this doctrine, by which we are enabled to compare curvilineal ſpaces in ſome of the more ſimple caſes.

This

This general principle may ſerve for demonſtrating many other propoſitions, beſides the elementary theorems already mentioned. For example, let ADB be a ſemicircle deſcribed on the diameter AB, AEB a ſemielliplſe deſcribed on the ſame right line as its tranſverſe axis; let AFGB be any polygon deſcribed in the ſemicircle; and let FN, GM, perpendicular to AB, meet the ſemielliplſe in H and K, and the axis in N and M. Becauſe any ordinate of the circle is to the ordinate of the ellipſe on the ſame point of the axis as the tranſverſe axis is to the conjugate, it follows, that the triangle ANF is to the triangle ANH, the trapezium FNMG to the trapezium HNMK, the triangle GMB to the triangle KMB, and the whole polygon AFGB to the polygon AHKB, in the ſame conſtant ratio of the tranſverſe to the conjugate axis. Biſect the arch FG in D, and the triangle FDG will be greater than half the ſegment in which it is inſcribed. Let the ordinate DI meet the ellipſe in E, and the triangle HEK will be alſo greater than half the elliptic ſegment in which it is inſcribed; for it is obvious that the right lines GF, KH produced meet the axis in the ſame point R, and that the tangents at D and E meet it in the ſame point T; conſequently the tangent DT being parallel to the chord FG, the tangent of the ellipſe at E is parallel to HK, and the triangle HEK is greater than half the ſegment HEK. The polygons therefore AFGB, AHKB, by continually biſecting the circular arches, may approach to the areas of the ſemicircle and ſemielliplſe, ſo as to differ from them by leſs than any aſſignable meaſure. Hence if the right line AD in the preceeding article repreſent the area of the ſemicircle, AB the area of the ſemielliplſe, AQ the polygon inſcribed in the ſemicircle, AP the correſponding polygon inſcribed in the ſemielliplſe; it will appear, in the ſame manner, that the ſemicircle muſt be to the ſemielliplſe in the ſame ratio that is the conſtant proportion of thoſe inſcribed polygons, *viz.* that of the tranſverſe axis of the ellipſe to the conjugate axis.

FIG. I.

We have given this demonſtration a little different from that of ARCHIMEDES in his fifth propoſition of *conoids* and *ſpheroids*, that the ſame proportion might appear to be the ratio of the area

rea NFDGM in the circle to the area NHEKM in the ellipſe. From which it follows, that if C be the common center of the circle and ellipſe, the triangles CFN, CHN and the triangles CGM, CKM being to each other as the tranſverſe axis is to the conjugate, the ſector CFDG in the circle muſt be to the ſector CHEK in the ellipſe in the ſame proportion. Let the diameter CE meet its ordinate HK in P, and CP being to CE as CR is to CT, or as CQ is to CD; it appears, that when the ratio of CP to the ſemidiameter of the ellipſe CE is given, then CQ is given, and therefore the ſector CFDG is of a determined magnitude; conſequently the elliptic ſector CHEK, and the ſegment HEK, muſt each be of a determined invariable magnitude in the ſame ellipſe, when the abſciſs CP is in a given ratio to the ſemidiameter CE. The triangle CHK, and the trapezium CHtK (formed by the ſemidiameters CH, CK and the tangents Ht, Kt,) are alſo given in magnitude when this ratio of CP to CE, or of CE to Ct, is given.

In general, if upon any diameter produced without the ellipſe, any number of points be taken, on the ſame or on different ſides of the center, at diſtances from it that are each in ſome given ratio to that diameter; and from theſe points tangents be drawn to the ellipſe in any one certain order; the polygon formed by theſe tangents is always of a given magnitude in a given ellipſe, and is equal to a polygon deſcribed by a ſimilar conſtruction about a circle, the diameter of which is a mean proportional betwixt the tranſverſe and conjugate axis of the ellipſe. The polygon inſcribed in the ellipſe by joining the points of contact, and the ſectors bounded by the ſemidiameters drawn to theſe points, are alſo of given or determined magnitudes; and the Parts of any tangent intercepted betwixt the interſections of the other tangents with it, or betwixt theſe interſections and the point of contact, are always in the ſame ratio to each other in the ſame figure. There is an analogous property of the other conic ſections.

When ARCHIMEDES demonſtrated, that the area of a circle is equal to a triangle upon a baſe equal to the circumference of the circle, of a height equal to the radius, it was not by ſup-

poſing

Pl. 1. Pa. 8.

Fig. 1.

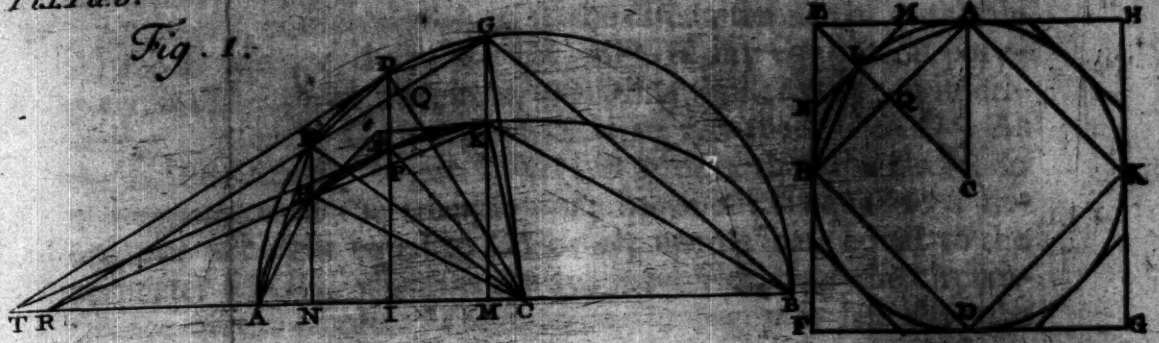


Fig. 2.

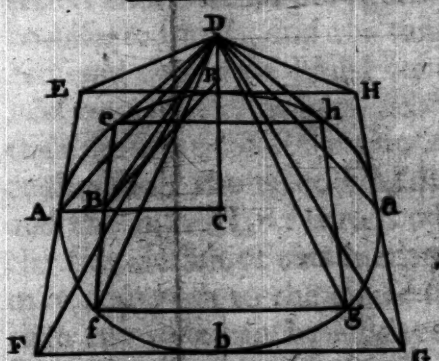


Fig. 4.

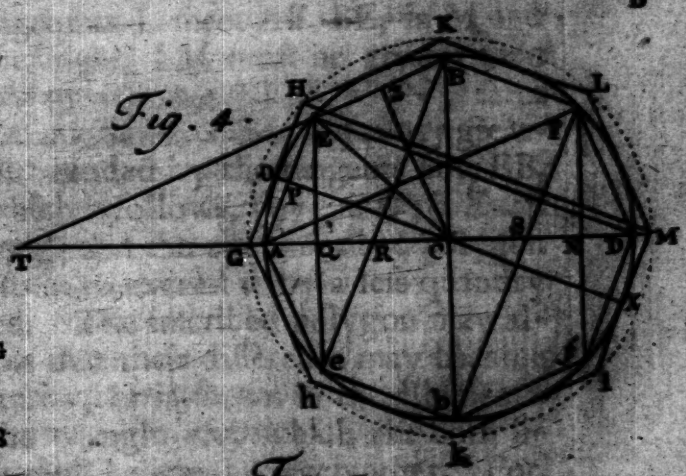


Fig. 3.

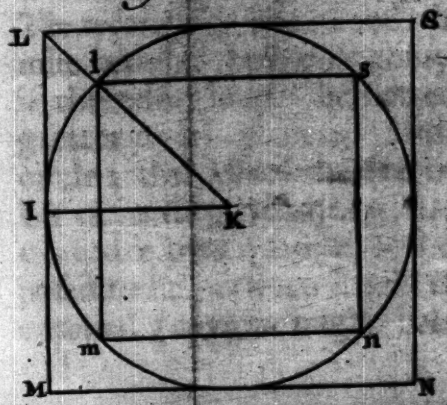
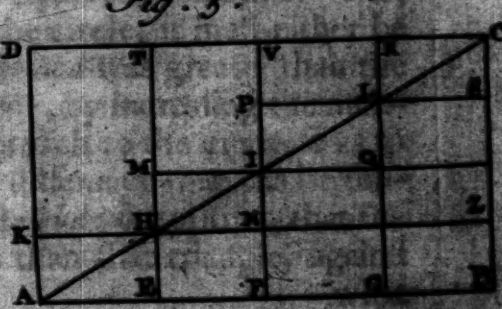
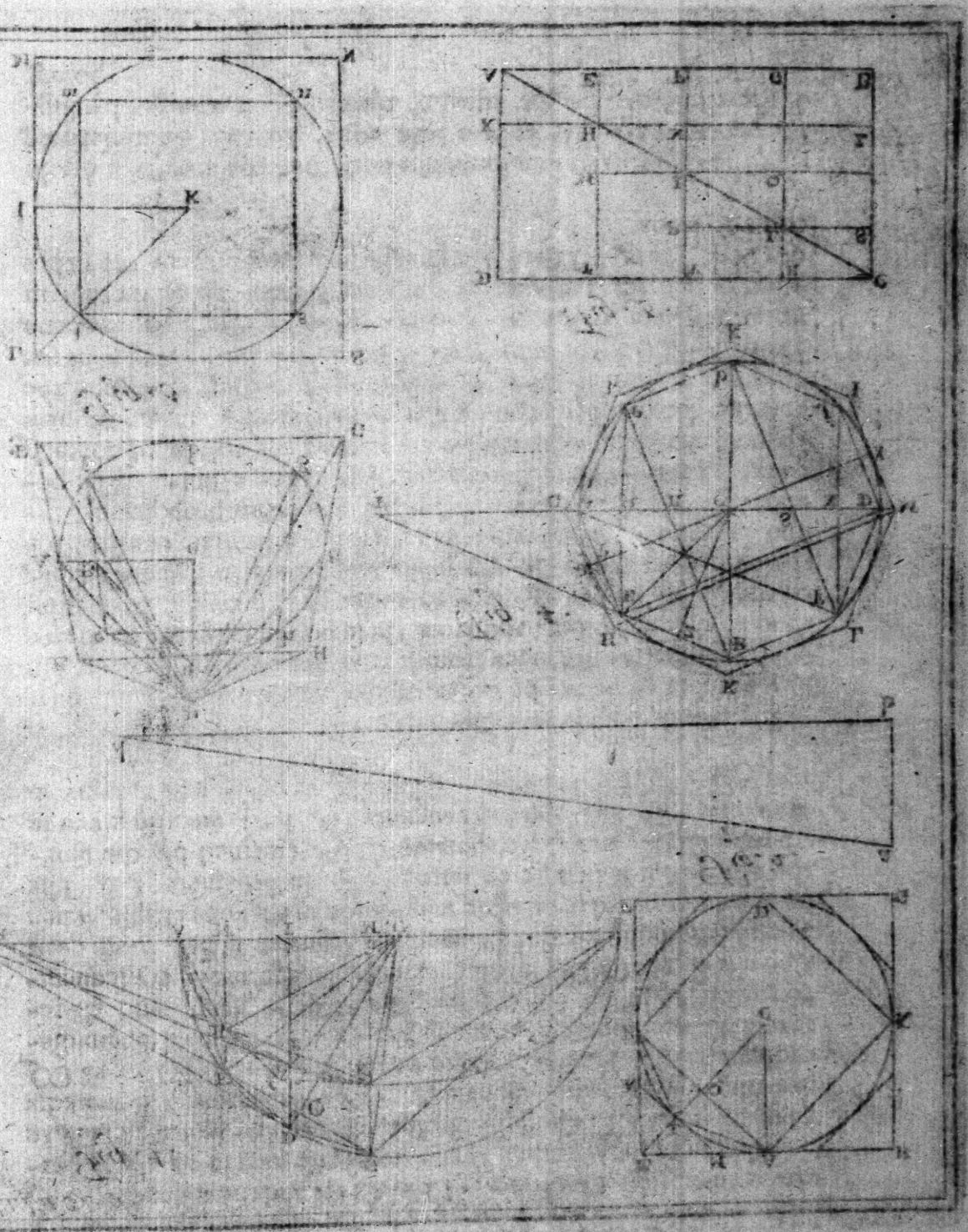


Fig. 5.





posing it to coincide with a circumscribed equilateral polygon of an infinite number of sides, but in a more accurate and unexceptionable manner. Let *bd*, the base of the right-angled triangle *abd*, be supposed equal to the circumference of the circle ABD, *ab* equal to the radius CA, EFGH any equilateral polygon described about the circle, ABDK a similar polygon inscribed in it, and let CQ perpendicular to AB meet it in Q. As the circumscribed polygon EFGH is greater than the circle, so it is greater than the triangle *abd*; because it is equal to a triangle of a height equal to CA or *ab*, upon a base equal to the perimeter EFGH, which is always greater than *bd* the circumference of the circle. The inscribed polygon is less than the circle; and it is also less than the triangle *abd*, because it is equal to a triangle of a height equal to CQ (which is less than CA or *ab*) upon a base equal to its perimeter ABDK which is less than the circumference of the circle *bd*. Therefore the circle and the triangle *abd* are both constantly limits betwixt the external and internal polygons EFGH, ABDK. Let the arch AB be bisected in L, and the tangent at L meet AE, BE, in M and N; and the angle ELM being right, EM must be greater than LM or AM, the triangle ELM greater than ALM, EMN greater than the sum of the triangles ALM, BLN, and consequently greater than half the space EALB bounded by the tangents EA, EB, and the arch ALB. From which it follows, (by the 1. 10. Eucl. the foundation of this method,) that the circumscribed polygon may approach to the circle so as to exceed it by a less quantity than any that can be assigned. The inscribed polygon may also approach to the circle so as that their difference may become less than any assignable quantity, as is shewn in the Elements. Therefore the circle and the triangle *abd*, which are both limits betwixt these polygons, must be equal to each other. For, if the triangle *abd* be not equal to the circle, it must either be greater or less than it. If the triangle *abd* was greater than the circle; then, since the external polygon, by increasing the number of its sides, might be made to approach to the circle so as to exceed it by a quantity less than any difference that can be supposed to be between it and the triangle *abd*; it follows, that the external polygon might become less than that triangle, against

FIG. 2.

what has been demonſtrated. If the triangle *abd* was leſs than the circle, then the inſcribed polygon, by being made to approach to the circle, might exceed that triangle: which, by what we have ſhewn, is alſo impoſſible.

In general, let any determined quantity AB be always a limit betwixt two variable quantities AP, AQ, which are ſuppoſed to approach continually to it and to each other, ſo that the difference of either from it may become leſs than any aſſignable quantity, or ſo that the ratio of AQ to AP may become leſs than any aſſignable ratio of a greater magnitude to a leſſer. Suppoſe alſo any other determined quantity *ab* to be always a limit betwixt the quantities *ap* and *aq*; and, *aq* being always equal to AQ or leſs than it, let *ap* be either equal to AP or greater than it. Then ſhall theſe limits AB and *ab* be equal to each other. For, if *ab* was equal to any quantity AE greater than AB, then, by ſuppoſing the ratio of AQ to AP to become leſs than that of AE to AB, the ratio of *aq* to AP would become leſs than the ratio of *ab* to AB: but ſince *aq* is always greater than *ab*, and AP leſs than AB, the ratio of *aq* to AP is always greater than the ratio of *ab* to AB; and theſe being repugnant, it follows, that *ab* is not greater than AB. If *ab* was equal to any quantity Ae leſs than AB, then, by ſuppoſing the ratio of AQ to AP to become leſs than that of AB to Ae, the ratio of AQ to *ap* (which, by the ſuppoſition, is either equal to AP, or greater than it) would become leſs than that of AB to *ab*: but ſince AQ always exceeds AB, and *ap* is always leſs than *ab*, the ratio of AQ to *ap* muſt be greater than that of AB to *ab*; and theſe being repugnant, it follows, that *ab* is not leſs than AB. Theſe limits therefore AB, *ab* are equal to each other.

A P B E Q
a p b q

In this manner ARCHIMEDES demonſtrates the fourteenth propoſition of the firſt book of his admirable treatiſe concerning the ſphere and cylinder. Let DA*ba* be an upright cone; D

FIG. 3.

the

the vertex, C the center and CA the radius of the base. Let KI be a mean proportional betwixt DA the side of the cone and CA the radius of its base; and the convex surface of the cone shall be equal to a circle of the radius KI. Let LMNS be any external equilateral polygon described about this circle, *lmns* a similar polygon inscribed in it; EFGH a similar polygon described about the base of the cone, *efgh* a similar polygon described in it: then the surface of the pyramid DEFGH, which is described about the convex surface of the cone, shall be to the polygon EFGH as DA is to CA, or as the square of KI is to the square of CA, or as the polygon LMNS is to the polygon EFGH; and therefore the surface of the external pyramid DEFGH, without the base, is equal to the polygon LMNS. Let CA, perpendicular to EF, meet *ef* in B; and BR, parallel to DA, meet CD in R: then the polygon *lmns* being to *efgh* as the square of KI is to the square of CA, that is, as DA is to CA, or as BR is to CB; and this being a less ratio than that of DB to CB, which is the same as the ratio of *Defgh*, the surface of the internal pyramid without its base, to the internal polygon *efgh*: it follows, that the surface of the internal pyramid without its base is greater than the polygon *lmns*. The external polygon LMNS is greater than the circle of the radius KI, the internal polygon is less than that circle, and the ratio of the external polygon to the internal may become less than any assignable ratio of a greater quantity to a lesser. The surface of the external pyramid DEFGH, without the base, is greater than the convex surface of the cone; and the surface of the internal pyramid *Defgh*, without the base, is less than it. Therefore by substituting, in the general demonstration of the last article, the circle of the radius KI in place of the quantity AB, the external polygon LMNS for AQ, the internal polygon *lmns* for AP, the convex surface of the cone for *ab*, the surfaces of the external and internal pyramids without their bases for *aq* and *ap*; it will appear, that the convex surface of the cone and the circle of the radius KI are equal to each other. We have given so full an account of this demonstration the rather that a part of it has been sometimes misrepresented.

It is much in the ſame manner that he demonſtrates the thirty firſt, fortieth and forty firſt, or, according to the celebrated Dr. BARROW's numbers, the thirty ſeventh, forty ninth and fiftieth of the ſame treatiſe: which are eſteemed by ſo good a judge, for the uſefulneſs of the propoſitions, the ſubtlety of the invention, and the elegance of the demonſtration, amongſt the moſt valuable diſcoveries in geometry. In the firſt of theſe he demonſtrates, that the ſurface of a ſphere is equal to four times the area of a great circle of the ſphere. Let AEBFD*fbe* be an equilateral polygon inſcribed in the generating circle, of a number of ſides that is any multiple of the number four; let GHKLM*klb* be a ſimilar polygon deſcribed about the circle, having its ſides parallel to thoſe of the internal polygon; and let E*e*, B*b*, F*f* meet the diameter AD in Q, C and N. Suppoſe the ſemicircle ABD, with the inſcribed and circumſcribed polygons, to revolve on the diameter AD: and the conical ſurface generated by the chord AE ſhall be equal to a circle of a radius that is a mean proportional betwixt AE and EQ; the ſurface generated by the chord EB ſhall be equal to a circle of a radius that is a mean proportional betwixt EB (or AE) and the ſum of the perpendiculars EQ and BC; and the ſurface generated by the whole perimeter of the internal polygon equal to a circle of a radius that is a mean proportional betwixt AE and the ſum of the perpendiculars E*e*, B*b* and F*f*. But ſuppoſing B*e* and F*b* to meet the diameter AD in R and S, the triangles DAE, AEQ, Q*e*R, RBC, C*b*S, SFN and DN*f* are ſimilar; and DE is to AE, EQ to AQ, *e*Q to QR, CB to CR, *b*C to CS, FN to SN, and *f*N to DN in the ſame ratio: and therefore the ſum of the perpendiculars E*e*, B*b*, F*f* is to the diameter AD as DE is to AE. From which it follows, that the ſurface generated by the revolution of the internal polygon AEBFD is equal to a circle of a radius that is a mean proportional betwixt AD, the diameter of the generating circle, and DE the right line drawn from one extremity of that diameter to the angle E that is next adjoining to its other extremity. In the ſame manner, the ſurface generated by the perimeter of the external polygon GHKLM is equal to a circle of a radius that is a mean proportional betwixt MH (which is equal to OX or AD) and GM the diameter

Fig. 4.

of

of the circle deſcribed about that polygon. Becauſe DE is leſs than AD, but GM greater than AD, it follows that the ſurface generated by the perimeter of the internal polygon is always leſs, and the ſurface generated by the perimeter of the external polygon is always greater than a circle of the radius AD which (becauſe AD is double of AC) is equal to four times the area of the generating circle. The ſurface of the ſphere is alſo itſelf always a limit betwixt theſe inſcribed and circumſcribed ſurfaces; and becauſe the ratio of theſe ſurfaces to each other is the duplicate of the ratio of GH to AE, or of CG to CA, it appears, that by increaſing the number of the ſides of the polygons the ratio of theſe ſurfaces may approach nearer to that of equality than any aſſignable ratio of inequality: therefore by ſubſtituting, in the general propoſition demonſtrated above, the ſurface of the ſphere in place of AB, the quadruple area of a great circle in place of *ab*, the ſurfaces generated by the circumſcribed and inſcribed polygons for AQ and AP, and ſuppoſing *aq* and *ap* reſpectively equal to AQ and AP; it will be evident that the ſurface of the ſphere and the quadruple area of its generating circle are equal to each other.

The ſurface generated by the perimeter AEBF, inſcribed in any arch AF, is equal to a circle of a radius that is a mean proportional betwixt DE and AN, which is leſs than AF the mean proportional betwixt AD and AN: and the ſurface generated by the perimeter GHKL being, for the ſame reaſon, equal to a circle of a radius that is a mean proportional betwixt AD and GN, which exceeds the chord AF; it follows, that a circle of the radius AF is always a limit betwixt the ſurfaces generated by the perimeters of the internal and external polygons AEBF and GHKL. But the portion of the ſpherical ſurface generated by the revolution of the arch AF is always a limit betwixt the ſame circumſcribed and inſcribed ſurfaces: and theſe being to each other in the duplicate ratio of GH to AE, or of CG to CA, and therefore in a proportion that may approach to the ratio of equality nearer than any ratio of inequality; it follows, that the portion of the ſpherical ſurface generated by the revolution of the arch AF is equal to the area of a circle deſcribed with

with a radius equal to the right line AF. The ſurfaces, therefore, generated by arches terminated at A are as the ſquares of their chords, or as their verſed ſines; and parallel planes which divide the diameter of a ſphere in equal parts, divide the ſurface of the ſphere into equal parts at the ſame time.

From theſe propoſitions, it is eaſy to ſee how he was able to compare the ſphere itſelf with the circumſcribed cylinder or inſcribed cone. Let CP, perpendicular from the center on AE, the ſide of the polygon, meet it in P: then the ſolid generated by the revolution of the triangle CAE being equal to a cone of the height CA upon a baſe equal to the circle generated by the perpendicular EQ, and this circle being to the ſurface generated by the revolution of the chord AE as the ſquare of EQ is to the rectangle contained under AE and EQ, that is, as EQ is to AE, or CP to CA; it follows, (ſuch cones being equal as have their heights and baſes reciprocally proportional,) that the ſolid generated by the revolution of the triangle CAE is equal to a cone upon a baſe equal to the ſurface generated by AE of a height equal to CP. Let BE produced meet the axis in T; and, for the like reaſon, the ſolids generated by the triangles CET, CBT ſhall be equal to cones upon baſes reſpectively equal to the ſurfaces generated by the revolution of the right lines ET and BT, of the height CZ equal to CP: ſo that their difference (or the ſolid generated by the triangle CEB) muſt be equal to a cone of the height CP upon a baſe equal to the ſurface generated by the right line EB. In like manner, the ſolid generated by the revolution of the whole inſcribed polygon is equal to a cone of the ſame height CP upon a baſe equal to the ſurface generated by the perimeter of the polygon, which we have ſhewn to be always leſs than four times the area of a great circle of the ſphere. For the ſame reaſon, the ſolid generated by the revolution of the external polygon GHKLM is equal to a cone of the height CO or CA, upon a baſe equal to the ſurface generated by the perimeter of that polygon, which, by what has been demonſtrated, is greater than four times the area of a great circle of the ſphere. A cone, therefore, upon a baſe equal to four times the area of a great circle of the ſphere, of an altitude

equal

equal to the radius, is always a limit betwixt the ſolids generated by the external and internal polygons. The ſphere itſelf is alſo a limit betwixt theſe ſolids: and their proportion being the triplicate of that of GH to AE, or of CG to CA, which may approach nearer to a ratio of equality than any aſſignable ratio of inequality; it follows, that the ſphere is equal to ſuch a cone, or to four times a cone that has its baſe equal to the generating circle and its height equal to the radius; and that the ſphere is to the circumſcribed cylinder as two is to three. The ſector of the ſphere generated by the revolution of the ſector of the circle CAE is ſhown, in like manner, to be equal to a cone upon a baſe equal to the portion of the ſpherical ſurface generated by the arch AE (or the circle deſcribed with the radius AE,) of a height equal to CA the radius of the ſphere. Theſe are the thirty ſecond and forty ſecond propoſitions of the ſame excellent treatiſe.

If we ſuppoſe the ſemicircle and ſemiellipſe to revolve on their common axis AB, and to generate a ſphere and ſpheroid; then the cone generated by the triangle AFN ſhall be to the cone generated by the triangle AHN in the duplicate ratio of FN to HN, or of the axis AB to its conjugate; becauſe cones of the ſame height are as their baſes. And in the ſame manner it appears, that the ſolid generated by the revolution of any polygon AFGB, inſcribed in the circle, is to the ſolid generated by the revolution of the correſponding polygon AHKB, inſcribed in the ellipſe, as the ſquare of the axis AB is to the ſquare of its conjugate. The ſolid generated by the triangle TID is to the ſolid generated by the triangle TIE as the ſquare of ID to the ſquare of IE, or as the ſquare of the axis AB to the ſquare of its conjugate: and if any polygon be deſcribed about the circle, and a polygon be alſo deſcribed about the ellipſe, ſo that the points of contact in the circle and ellipſe be always in the ſame perpendiculars to the axis, the ſolids generated by theſe polygons ſhall be to each other in the ſame proportion. By ſuppoſing the circumſcribed and inſcribed polygons in the circle to be equilateral, and to have their correſponding ſides parallel, it may be demonſtrated, that the ſphere is to the ſpheroid in the dupli- FIG. 1.

duplicate ratio of the axis AB to its conjugate, and that the ſolids generated by the ſpaces HNMG, HNMK are in the ſame proportion. The ſpheroid, therefore, is quadruple of the cone of a height equal to half the axis AB, on a baſe equal to the circle deſcribed upon the conjugate axis as its diameter.

As the ſegment of the ellipſe HEK, and the triangle HEK, are of a determined invariable magnitude in a given ellipſe, when the ratio of CP to CE is given; ſo the portion of the ſpheroid (generated by the ellipſe revolving on its axis AB) which is cut off by a plane through HK perpendicular to the plane AEB is of a determined magnitude, when the ratio of CP to CE is given. A cone upon the ſame baſe with this portion, that has its vertex in E, or in *t*, is alſo of an invariable magnitude, when that ratio is given; and there is a general property of the circumſcribed and inſcribed ſolids analogous to that above mentioned of the circumſcribed and inſcribed polygons.

Archimedes takes a different way for comparing the ſpheroid with the cone and cylinder, that is more general, and has a nearer analogy to the modern methods. He ſuppoſes the terms of a progreſſion to increaſe conſtantly by the ſame difference, and demonſtrates ſeveral properties of ſuch a progreſſion relating to the ſum of the terms and the ſum of their ſquares; by which he is able to compare the parabolic conoid, the ſpheroid and hyperbolic conoid, with the cone, and the area of his ſpiral line with the area of the circle. There is an analogy betwixt what he has ſhewn of theſe progreſſions, and the proportions of figures demonſtrated in the elementary geometry; the conſideration of which may illuſtrate his doctrine, and ſerve perhaps to ſhew that it is more regular and compleat in its kind than ſome have imagined *. The relation of the ſum of the terms to the quantity that ariſes by taking the greateſt of them as often as there are terms, is illuſtrated by comparing the triangle with the parallelogram of the ſame height and baſe; and what he has demonſtrated of the ſum of the ſquares of the terms

* See the preface to the *Analyſe des infiniment petits.*

com-

compared with the ſquare of the greateſt term, may be illuſtrated by the proportion of the pyramid to the priſm, or of the cone to the cylinder, their baſes and heights being equal; and by the ratios of certain fruſtums or portions of theſe ſolids, which may be deduced from the elementary propoſitions.

The baſe AB of the rectangle ABCD being divided into any number of equal parts, AE, EF, FG, GB, and the perpendiculars ET, FV, CR being drawn meeting AC in H, I, L, and CD in T, V and R; the rectangles AH, EI, FL, GC may repreſent a progreſſion of terms that conſtantly increaſe by a common difference equal to the leaſt term AH. The ſum of theſe rectangles form the figure AKHMIPLRCB which is deſcribed about the triangle ACB, and is always greater than the triangle: Their ſum without the greateſt term GC is equal to the figure EHNIQLSB, which is inſcribed in the triangle, and therefore is always leſs than it. The rectangle ABCD is the ſum of as many terms equal to the greateſt GC, as there are terms. And as the triangle ACB, or one half of the rectangle ABCD, is always a limit betwixt theſe circumſcribed and inſcribed figures; ſo, in general, if the greateſt term of any progreſſion of this kind be taken as often as there are terms, then one half of the quantity that is thus produced ſhall be always a limit betwixt the ſum of all the terms and this ſum without the greateſt term: which coincides with the firſt propoſition of the treatiſe of conoids and ſpheroids.

Suppoſe the ſame figure to revolve upon the axis AB, and while the triangle ABC generates a cone equal to the third part of the cylinder deſcribed by the rectangle ABCD, the rectangles AH, EI, FL, GC generate cylinders of equal heights; which are therefore as the ſquares of the right lines EH, FI, GL, BC, that increaſe conſtantly by the ſame difference equal to the firſt term EH. The ſum of theſe cylinders, or the ſolid generated by the figure AKHMIPLRCB circumſcribed about the triangle ABC, always exceeds the cone generated by that triangle: Their ſum without the cylinder generated by the rectangle GC, being equal to the ſolid generated by the inſcribed figure EHNIQLSB, is always leſs than the ſame cone. The cylinder

generated by the rectangle ABCD is the ſum of as many cylinders equal to that generated by GC, as there are terms. And as the cone generated by the triangle ABC, or one third part of the cylinder generated by ABCD, is always a limit betwixt the ſolids generated by the figures AKHMIPLRCB, EHNIQLSB; ſo, in general, one third part of the quantity that is produced by taking the ſquare of the greateſt term of any progreſſion of this kind as often as there are terms, is always a limit betwixt the ſum of the ſquares of all the terms and the ſame ſum without the ſquare of the greateſt term. ARCHIMEDES demonſtrates this propoſition in a general way, and by it the proportion of the area of his firſt ſpiral line to the circumſcribed circle. But PAPPUS * makes uſe of the ratio of the cone to the cylinder for the ſame purpoſe; and ſince we have the example of ſo accurate an author, we ſhall proceed to deſcribe ARCHIMEDES's other theorems of this kind in the ſame manner.

The eleventh propoſition of the treatiſe concerning ſpiral lines appears from comparing the cylinder generated by the rectangle EBCT with the fruſtum of a cone generated by the trapezium EBCH, and with the ſolids generated by the figures EMIPLRCB, EHNIQLSB, the former of which is always greater than that fruſtum, and the latter leſs than it. For the cylinder generated by the rectangle EBCT is to the fruſtum of a cone generated by the trapezium EBCH, as the ſquare of BC is to the rectangle contained under the right lines BC and EH added to one third part of the ſquare of the difference of theſe lines. The demonſtration of which we ſhall ſubjoin, the rather that this propoſition will be of uſe afterwards.

The reſt of the conſtruction remaining as in the fifth figure, let ES parallel to AC, and HZ parallel to AB, meet BC in S and Z; and, ſuppoſing BX to be a mean proportional betwixt BC and EH, or BZ, complete the parallelogram EBXY. Then let the rectangles EBCT, EBXY, by revolving on the axis AB, generate the cylinders TC*ct*, YX*xy*, and the triangles ABC,

FIG. 6. Plate 2.

* Collect. Mathem. lib. 4. prop. 21.

AEH,

AEH, AET, EBC, EBS, EBX, EBZ generate the cones AC*c*, AH*b*, AT*t*, EC*c*, ES*s*, EX*x*, EZ*z*; and the ſolid generated by the parallelogram ESCH ſhall be equal to the cylinder YX*xy*: as may be ſhewn from this known propoſition, That the cube of BC exceeds the two cubes of BZ and CZ (or BS) by three times the parallelepipedon upon the ſquare of BX of the height CZ: From which it follows, that the cone AC*c* exceeds the cones AH*b*, ES*s* by a ſolid equal to the cylinder YX*xy*. The ſame theorem may be demonſtrated from the elementary propoſitions thus: The ſolid generated by the parallelogram ESCH is the ſum of theſe which are generated by the triangles ESC and ECH. That which is generated by ESC is equal to the exceſs of twice the cone EX*x* above the cone EZ*z*, becauſe the ſquare of BC (7. 2. Eucl.) added to the ſquare of BZ is equal to two rectangles under BC and BZ (or twice the ſquare of BX) added to the ſquare of CZ or BS; and therefore the difference of the cones EC*c*, ES*s* (or the ſolid generated by ESC) is equal to the exceſs of twice the cone EX*x* above the cone EZ*z*. The ſolid generated by the triangle ECH is equal to the exceſs of the ſolid generated by AEC (or the cone AT*t*) above the cone AH*b*. But the ſum of the ſquares of BC and BX is to the ſum of the ſquares of BX and BZ as BC is to BZ, or as AB is to AE; and therefore (17. 5. Eucl.) the difference of the ſquares of BC and BZ is to the ſum of the ſquares of BX and BZ as EB is to AE: from which it follows, that the difference of the cones AT*t*, AH*b* (or the ſolid generated by ECH) is equal to the ſum of the cones EX*x* and EZ*z*. Therefore the ſolid generated by the parallelogram ESCH is equal to thrice the cone EX*x*, or to the cylinder XY*yx*; and the fruſtum HC*ch* is equal to that cylinder added to the cone ES*s*: ſo that the cylinder TC*ct* is to the fruſtum HC*ch* as the ſquare of BC is to the ſquare of BX (or the rectangle under BC and EH) added to one third part of the ſquare of BS, or CZ, the difference of BC and EH. It appears, therefore, that the cylinder generated by the rectangle EBCT is to the ſolid generated by the figure EMIPLRCB in a leſs proportion, and to the ſolid generated by the figure EHNIQLSB in a greater proportion, than the ratio of the ſquare of BC to the rectangle under the right

lines BC and EH added to one third part of the ſquare of the difference of theſe right lines. In general, the quantity that is produced by taking the ſquare of the greateſt term (in ſuch a progreſſion as we deſcribed) one time leſs than the number of terms, is to the ſum of the ſquares of the terms without the ſquare of the leaſt of them in a leſs proportion, and to the ſum of their ſquares without the ſquare of the greateſt in a higher proportion, than the ratio of the ſquare of the greateſt term to the rectangle under the extreme terms added to the third part of the ſquare of their difference.

In like manner, the third propoſition of the treatiſe concerning conoids and ſpheroids appears by comparing the ſolid generated by the rectangle HZCT with thoſe generated by the triangle HZC and the figures HMIPLRCZ, NIQLSZ, revolving about the axis AB; ſuppoſing EB to be divided into any equal parts EF, FG, GB. For the ſolid generated by any rectangle NM being to the ſolid generated by the rectangle NL as the difference of the ſquares of FI and EH to the difference of the ſquares of GL and EH, or as the rectangle contained under NI and the ſum of 2EH and NI (6. 2. Eucl.) is to the rectangle under WL and the ſum of 2EH and WL, the ſolids generated by the rectangles NM, WP, ZR increaſe in the ſame proportion as the ſpaces deſcribed in that propoſition, which are ſuppoſed to be applied on a given line, (as 2EH,) and to exceed by ſquares that have their ſides (as NI, WL, ZC) conſtantly increaſing by a difference equal to the ſide of the firſt exceſs. But the ſolid generated by the triangle HZC is equal to the ſolid generated by the rectangle HX added to the cone ES*s*, by what has been demonſtrated; and therefore the ſolid generated by the rectangle HZCT is to the ſolid generated by the triangle HZC as the difference of the ſquares of BC and BZ (or the rectangle CZ*c*) is to the difference of the ſquares of BX and BZ (or the rectangle CZB) added to one third part of the ſquare of BS, or CZ; that is, as Z*c*, or the ſum of BC and BZ, is to the ſum of BZ and one third part of CZ; and therefore as the ſum of 2EH and CZ is to the ſum of EH and one third part of CZ. From which it follows, that the ſolid generated by the

the rectangle HC is to that which is generated by the figure HMIPLBCZ in a leſs proportion, and to the ſolid generated by NIQLSZ in a greater proportion, than that ratio. From this and the preceeding theorems ARCHIMEDES deduces many propoſitions relating to conoids and ſpheroids, and various portions of theſe ſolids: the moſt conſiderable of which may be demonſtrated in the following manner, that differs little from his own method, and may poſſibly be the ſame by which he diſcovered them.

Let AIC be a parabola, A the vertex, AB a part of the axis, BC an ordinate; complete the rectangle ABCD, and the conoid generated by the parabolic area AICB about the axis AB ſhall be equal to one half of the cylinder generated by the rectangle ABCD. By continually biſecting the parts of the axis AB let it be divided into the equal parts AE, EF, FG, GB; and let the ordinates EH, FI, GL meet the curve in H, I, L, the right line DC in T, V, R, and the right line AC in X, Y and Z. Complete the rectangles AH, FH, EI, GI, FL, GS, as alſo the rectangles AX, EY, FZ, GC, E*n*, F*q* and G*s*; and the cylinders generated by AH, EI, FL, GC, having equal altitudes, are as their baſes, or as the ſquares of the ordinates EH, FI, GL BC, and therefore as the parts of the axis AE, AF, AG, AB, or as the rectangles AX, EY, FZ, GC. From which it follows, that the cylinder generated by the rectangle ABCD, the ſolids generated by the circumſcribed figure AKHMIPLRCB and the inſcribed figure EHNIQLSB, are in the ſame proportion to each other as the rectangle ABCD and the figures A*k*X*m*Y*p*ZRCB, EX*n*Y*q*Z*s*B; and conſequently one half of the cylinder generated by ABCD is always a limit betwixt the ſolid circumſcribed about the conoid and that which is deſcribed in it. The conoid itſelf is alſo always a limit betwixt the ſame circumſcribed and inſcribed ſolids; and the difference of theſe ſolids being equal to the cylinder generated by the rectangle GC, which, by continually biſecting the parts of the axis, may become leſs than any ſolid that can be aſſigned: it follows, that the conoid generated by the parabolic area AICB is equal to half the cylinder generated by the rectangle ABCD. If a parabolic conoid be cut by planes parallel to each other, but oblique to the axis, the ſections are FIG. 7.

ſimilar

ſimilar ellipſes; and any portion of the ſolid cut off by ſuch a plane is ſhewn, in like manner, to be equal to one half of a cylinder of the ſame baſe and altitude.

FIG. 8. Let CA, CB be the ſemiaxes of an ellipſe AIB; complete the rectangle CADB, and join CD. Let CA be divided, by a continual biſection, into the equal parts AE, EF, FG, GC; draw the ordinates EH, FI, GL meeting BD in T, V, R, and CD in X, Y, Z; complete the rectangles AH, FH, EI, GI, FL, CL, and the rectangles DX, VX, TY, RY, VZ, BZ. Then, becauſe the ſquare of any ordinate EH is to the difference of the ſquares of CA and CE as the ſquare of CB is to the ſquare of CA, and the difference of the ſquares of AD (or ET) and EX is to the difference of the ſquares of CA and CE in the ſame proportion; it follows, that the ſquare of EH is equal to the difference of the ſquares of ET and EX, and that the circle deſcribed by EH is equal to the difference of the circles deſcribed by ET and EX: ſo that the cylinder generated by the rectangle AEHK muſt be equal to the ſolid generated by the rectangle DTX*k*. In like manner, the ſolid generated by the circumſcribed figure AKHMIPLRBC is equal to that which is generated by the figure D*k*X*m*Y*p*ZGCB, which always exceeds the ſolid that is generated by the triangle CDB; and the ſolid generated by the inſcribed figure EHNIQLSC is equal to that which is generated by the figure TX*n*Y*q*Z*s*B, which is always leſs than the ſolid generated by that triangle. Therefore the ſolid generated by the triangle CDB and the part of the ſpheroid which is generated by the elliptic area AIBC are conſtantly limits betwixt the ſame circumſcribed and inſcribed ſolids; and the difference of theſe ſolids being equal to the cylinder generated by the rectangle BG, which, by continually biſecting the parts of the axis, may become leſs than any ſolid that can be aſſigned; it appears, that the ſolids generated by the elliptic area AIBC and the triangle DCB are equal, and that the portion of the ſpheroid generated by the elliptic area AEH is equal to the ſolid generated by the triangle DTX revolving about the axis AC. But the ſolid generated by the triangle DX*k*, by what has been demonſtrated, is to the ſolid generated by the rectangle DTX*k* (or the cylinder generated

generated by AEHK) as the ſum of EX and one third part of TX is to the ſum of 2EX and TX, or as the ſum of CA and 2CE is to the triple ſum of CA and CE; and therefore the portion of the ſpheroid generated by the elliptic area AEH is to the inſcribed cone generated by the triangle AEH as the ſum of 2CA and CE is to the ſum of CA and CE.

In the ſame manner, CA being any ſemidiameter of the generating ellipſe, B*b* the conjugate diameter, D*d* a tangent at A, BD and *bd* parallel to CA, ſuppoſe a cylinder BD*db* to be deſcribed about half the ſpheroid AB*kn* touching it in the circumference of the ſection B*kbn* perpendicular to the plane AB*b*, and ſuppoſe the cone CD*rdu* to be inſcribed in this cylinder. Then, if any plane cut BA*b* perpendicularly in the ordinate H*h*, (that meets BD, *bd*, CD, *cd* in T, *t*, X and *x*,) the ſections of the ſpheroid, cylinder and cone made by this plane being ſimilar ellipſes, and the ſquare of EH being equal to the exceſs of the ſquare of ET above the ſquare of EX; it follows, that the ellipſe which is the ſection of the ſpheroid with that plane, is equal to the difference of the ellipſes that are the ſections of the cylinder and cone made by the ſame plane. From which it may be demonſtrated, as in the preceeding article, by continually biſecting the parts of CA, and by circumſcribing and inſcribing cylinders of equal heights about the ſpheroid and cone, that the portion of the ſpheroid AH*b* is equal to the exceſs of the fruſtum of the cylinder DT*td* above the fruſtum of the cone DX*xd*; and that the portion of the ſpheroid AH*b* is to the inſcribed cone AH*b* of the ſame baſe and altitude as the ſum of 2CA and CE is to the ſum of CA and CE. FIG. 9.

Let AIB be an hyperbola, O the center, AC a part of the axis, A the vertex, AD a tangent meeting the aſymptote OD in D, CB an ordinate meeting OD in *e* and D*d* parallel to the axis in *d*; let *ef*, B*b* parallel to the axis meet AD in *f* and *b*, and the reſt of the conſtruction be ſimilar to that of the eighth figure. The ſquare of any ordinate EH being equal to the difference of the ſquares of EX and ET, the circle deſcribed by EH is equal to the difference of the circles deſcribed by EX and ET, FIG. 10.

the

the cylinder generated by the rectangle AEHK equal to the ſolid generated by the rectangle DTX*k*; and, in like manner, the ſolids generated by the whole circumſcribed and inſcribed figures AKHMIPLRBC, EHNIQLSC are reſpectively equal to the ſolids generated by the figures *DkXmYpZged*, TX*n*Y*q*Z*sd*. From which it follows, that the ſolid generated by the triangle D*de* is always a limit betwixt the ſolid generated by the figure deſcribed about the hyperbolic area, and that which is generated by the figure deſcribed in it. The portion of the conoid generated by that area is always a limit betwixt the ſame ſolids: and the difference of theſe ſolids being equal to the cylinder generated by the rectangle GB, which, by continuing to biſect the parts of the axis, may become leſs than any ſolid that can be aſſigned; it appears that the portion of the conoid generated by the area ABC is equal to the ſolid generated by the triangle D*de*. But this ſolid, by what has been demonſtrated, is to the ſolid generated by the rectangle D*def*, or cylinder generated by the rectangle C*b*, as the ſum of 2AD and C*e* is to the triple ſum of AD and C*e*; and therefore the conoid generated by the hyperbolic area ABC is to the inſcribed cone generated by the triangle ABC as the ſum of 2AD and C*e* is to the ſum of AD and C*e*, or as the ſum of 2OA and OC is to the ſum of OA and OC. Any portion of the conoid cut off by a plane oblique to the axis, is to the inſcribed cone in a like proportion; as may be demonſtrated much in the ſame manner.

The following general theorem comprehends the preceeding propoſitions and ſeveral others of the ſame kind. It contains a property which extends to all the ſolids that can be generated by any conic ſection revolving upon its axis, including the ſphere and cone, and may be of uſe in menſuration. A fruſtum or portion of any ſuch ſolid terminated by any two parallel planes, and a cylinder of the ſame height with the fruſtum upon a baſe equal to the ſection of the ſolid made by a parallel plane that biſects the altitude of the fruſtum, differ from each other always by the ſame magnitude, in the ſame or in ſimilar ſolids, when the inclination of the planes to the axis and the altitude of the fruſtum are given. In the parabolic conoid this difference

vaniſhes, the fruſtum being always equal to a cylinder of the ſame height upon the ſection of the conoid that biſects the altitude of the fruſtum and is parallel to its baſes. In the ſphere, the fruſtum is always leſs than the cylinder by one fourth part of a right-angled cone of the ſame height with the fruſtum, or by one half of a ſphere of a diameter equal to that height: and this difference is always the ſame in all ſpheres whatſoever, when the altitude of the fruſtum is given. In the cone, the fruſtum always exceeds the cylinder by one fourth part of the content of a ſimilar cone that has the ſame height with the fruſtum. In the hyperbolic conoid, this exceſs is the ſame as in the cone generated by the triangle OC*e* formed by the axis OC the aſymptote O*e* and the perpendicular C*e*, the altitude of the fruſtums and the inclination of the axis to their baſes being the ſame in both. In the ſpheroid AB*kb*, the cylinder exceeds the fruſtum: and the difference betwixt them is the ſame as in the cone CD*rd*, the plane D*rd*, or B*kb*, being ſuppoſed parallel to thoſe which terminate the fruſtum. In different inclinations of thoſe planes, when the altitude of the fruſtum is given, that difference is reciprocally as the cube of the diameter B*b* which is the conjugate of CA, the axis of the fruſtum. But if the altitude of the fruſtum be alſo varied ſo as to be reciprocally proportional to the diameter B*b*, then the difference betwixt the fruſtum and cylinder ſhall be always of the ſame magnitude in the ſame ſpheroid or conoid. When the inclination of the axis of the ſolid to the planes that terminate the fruſtum is given, the difference betwixt the fruſtum and cylinder, in the ſame or in ſimilar ſolids, is as the cube of their common altitude.

FIG. 10.

FIG. 9.

The truth of this general theorem will eaſily appear from what has been demonſtrated of theſe ſolids, if we prove that it obtains in the cone. Reſuming, therefore, the conſtruction of the ſixth figure, let EB the altitude of the fruſtum generated by the trapezium EBCH be biſected in *f*; let *fa* parallel to BC meet AC in *a*, *eb* parallel to AB paſſing through *a* meet BC and EH in *b* and *e*; and, BR being equal to *b*Z, or one half of CZ, complete the parallelogram EBRV. Then, ſince CZ is biſected in *b*, the rectangle CBZ (or the ſquare of BX) added

FIG. 11.

ded to the ſquare of *b*Z, or BR, is equal (6. 2. Eucl.) to the ſquare of B*b*; and the cylinder generated by the rectangle B*e* is therefore equal to the ſum of the cylinders generated by the rectangles BY and BV. The fruſtum of a cone generated by the trapezium EBCH is equal to the cylinder generated by the rectangle BY added to the cone generated by the triangle EBS, by what has been demonſtrated; and therefore this fruſtum exceeds the cylinder generated by the rectangle B*e* by the exceſs of the cone generated by the triangle EBS above the cylinder generated by the rectangle BV; that is, (becauſe BR is one half of BS,) by one fourth part of the cone generated by the triangle EBS. But this exceſs is always of the ſame magnitude, in the ſame or in ſimilar cones, when EB the altitude of the fruſtum is given: Therefore, if a cone is cut by three planes perpendicular to its axis at equal diſtances from each other, the fruſtum comprehended betwixt the firſt and third plane exceeds the cylinder of the ſame height upon the middle ſection as its baſe, by a ſimilar cone of a height equal to the altitude of the fruſtum; which exceſs is always of the ſame magnitude, in the ſame or in ſimilar cones, when that height is given. In like
FIG. 12. manner, it may be demonſtrated, that if the parallel planes C*mc*, K*nk*, H*ub*, at equal diſtances from each other, be oblique to the axis of the cone, but perpendicular to the plane ACB*c* that paſſes through the axis in the right lines C*c*, K*k* and H*b*; and ES*xs* be a cone ſimilar to AC*mc*, of the ſame height with the fruſtum CH*bc*: then this fruſtum ſhall exceed the cylinder of an equal height on the baſe K*nk* by one fourth part of the cone ES*xs*.

FIG. 8. The fruſtum of the ſpheroid generated by the area EHLG is equal to the ſolid generated by the trapezium TXZR; and the cylinder generated by the rectangle EMQG is equal to the ſolid generated by the rectangle TR*qm*, by what has been demonſtrated above: Therefore the difference betwixt the fruſtum and cylinder in the ſpheroid is the ſame as in the cone generated by the triangle CAD, the altitudes of the fruſtums being equal. In the hyperbolic conoid generated by the area
FIG. 10. AIBC revolving on the axis AC, the fruſtum generated by the area

Plat Pa. 26

Fig: 6.

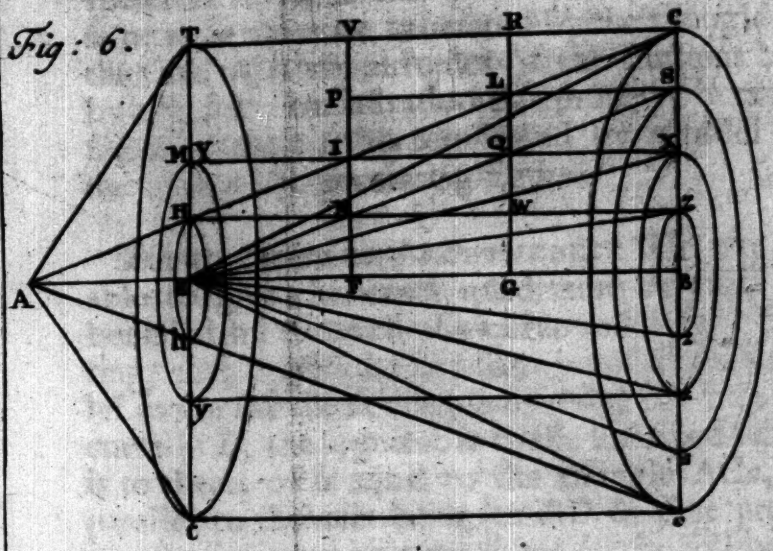


Fig. 7.

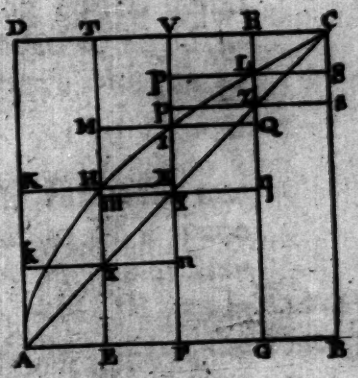


Fig. 10.

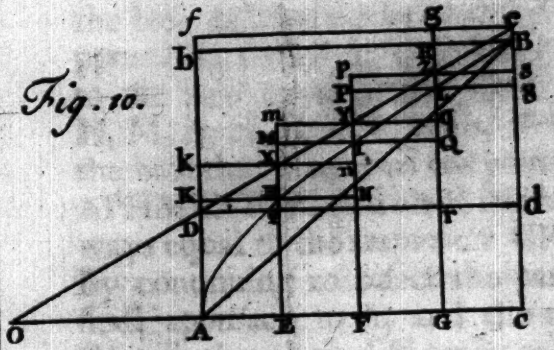


Fig. 8.

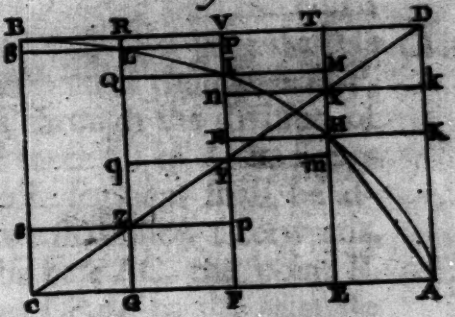


Fig. 9.

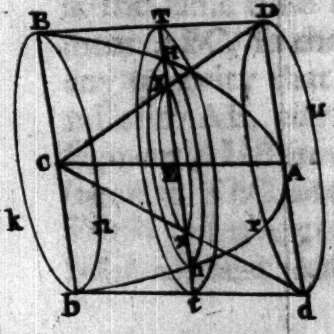


Fig. 11.

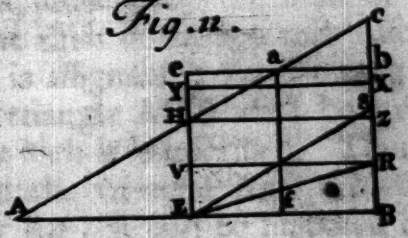
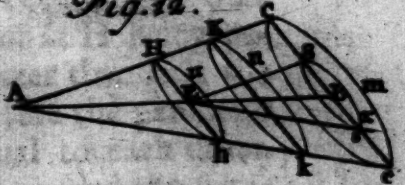
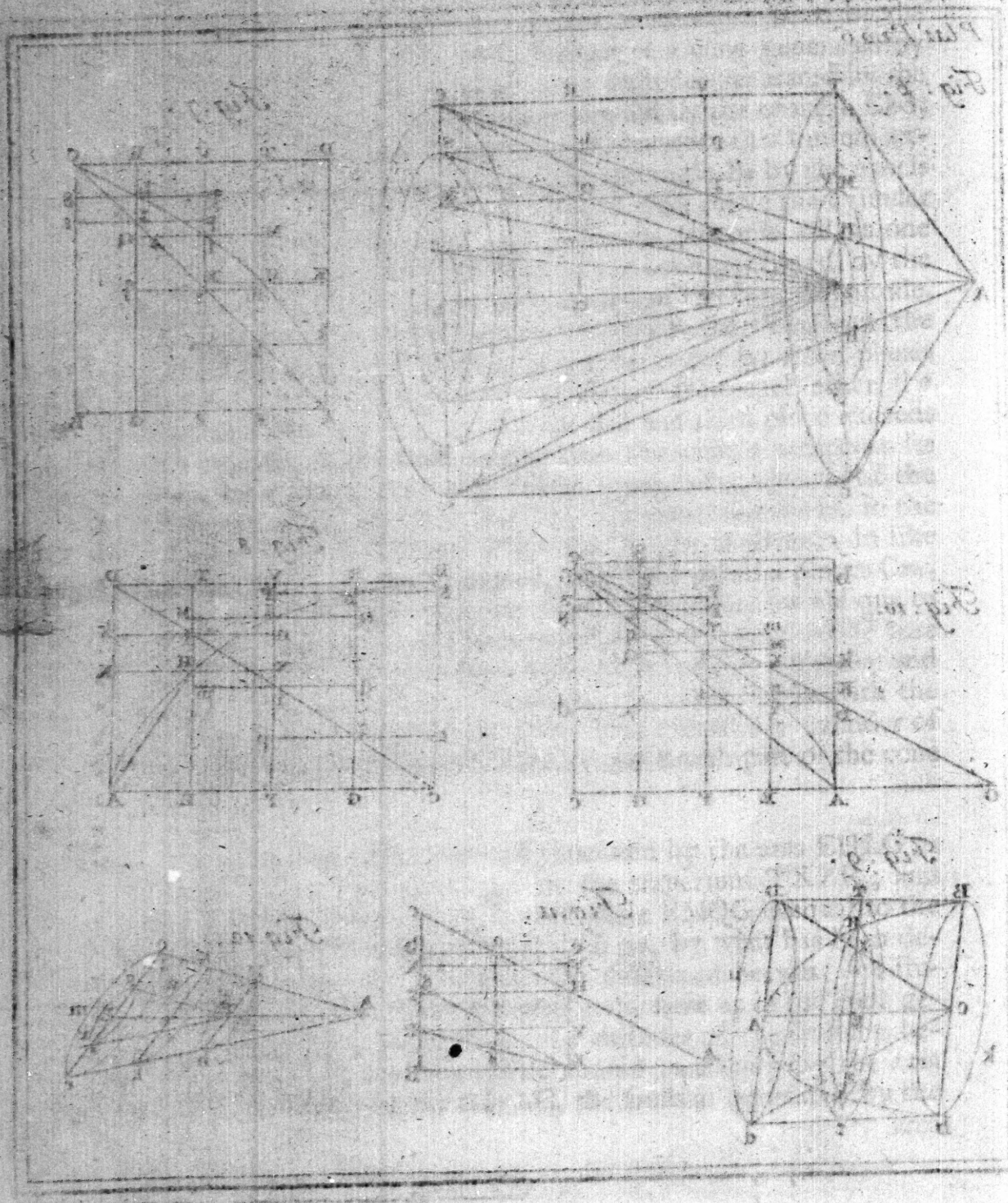


Fig. 12.





area EHLG is equal to the ſolid generated by the trapezium TXZ*r*, and the cylinder generated by the rectangle EMQG is equal to the ſolid generated by the rectangle *Trqm*; and therefore the difference betwixt rhe fruſtum and cylinder is the ſame as in the cone generated by the triangle OC*e* revolving on the ſame axis, the altitudes of the fruſtums being equal. And by a ſimilar demonſtration this property is extended to any fruſtums of theſe ſolids terminated by parallel planes oblique to the axis of the generating figure.

But to return to ARCHIMEDES: He was the firſt who was able to give the exact quadrature or menſuration of a ſpace bounded by the arch of a curve and a right line, by demonſtrating, that if ABC be any ſegment of a parabola, and DB paral- FIG. 13. lel to the axis of the figure biſecting the baſe AC in D meet the Plate 3. curve in B, the ſegment is to the inſcribed triangle ABC as four is to three, or is equal to the triangle AC*a*, the right line C*a* parallel to the axis being to BD in that proportion. His demonſtration may be repreſented in the following manner. Let the baſe AC by a continual biſection be divided into the equal parts AR, RG, GS, SD, DX, XE, EY, YC, and let parallels to the axis through the points of diviſion meet the curve in P, H, M, B, N, F, V. Then, if RP paſs through the diviſion of the baſe that is next to the point A, and meet A*a* in *d*, the figure APHMBNFVC inſcribed in the parabolic ſegment ſhall be always equal to the trapezium CR*da*, as we ſhall ſhew afterwards. By continuing to biſect the parts of the baſe AC, the point R ſhall approach to A, and the trapezium CR*da* to the triangle CA*a* ſo that their difference may become leſs than any quantity that can be aſſigned. The inſcribed polygon APHMBNFVC at the ſame time approaches to the area of the parabolic ſegment ABC, ſo that their difference may alſo become leſs than any quantity that is aſſignable; for the tangent at H being parallel to AB, any triangle AHB muſt be greater than half the ſegment AHB in which it is inſcribed. Therefore the parabolic ſegment ABC and the triangle A*a*C are equal to each other. For if the triangle A*a*C ſhould be ſuppoſed to exceed the parabolic ſegment ABC by any ſpace, as O; then, ſince by conti-

nuing the diviſions of the baſe AC the triangle AR*d* might become leſs than the ſpace O, the trapezium CR*da*, by approaching to the triangle CA*a* ſo as to differ from it by a ſpace leſs than O, might exceed the parabolic ſegment ABC; and the polygon APHMBNFVC, which is always equal to the trapezium CR*da*, might exceed the parabolic ſegment in which it is inſcribed: which is abſurd. If the parabolic ſegment ſhould be ſuppoſed to exceed the triangle AC*a* by any ſpace O; then, ſince by continuing to biſect the baſe AC the inſcribed polygon might approach to the parabolic ſegment ſo as to differ from it by a ſpace leſs than O, this polygon might exceed the triangle CA*a*. But the inſcribed polygon being always equal to the trapezium CR*da*, and the point R being always betwixt A and C, the polygon muſt be always leſs than the triangle CA*a*; and theſe are contradictory. Therefore the parabolic ſegment ABC and the triangle A*a*C are preciſely equal to each other, or the ſegment ABC is to the inſcribed triangle ABC as C*a* is to BD, that is, as four is to three.

We ſuppoſed, that the inſcribed polygon APHMBNFVC is always equal to the trapezium CR*da*. To complete the demonſtration, let DB biſecting AC, GH biſecting AD, RP biſecting AG meet A*a* in the points *b*, *c*, *d* reſpectively; let GH, RP meet AB, AH in L and *l*; and let HQ parallel to AC meet BD in Q, and P*q* parallel to AB meet GH in *q*. The triangle ABC is equal to the trapezium CD*ba*; becauſe the triangle AC*a* is to the triangle ABC as C*a* is to BD, or as four is to three: and the triangle AC*a* being quadruple of AD*b*, it is to the trapezium CD*ba* in the ſame proportion of four to three. The ſum of the triangles AHB, BFC is equal to the trapezium DG*cb*; becauſe the ſum of theſe triangles is to the triangle ABC as HL (or BQ) is to BD, or as the ſquare of HQ (or GD) is to the ſquare of AD; that is, as one is to four: and the trapezium DG*cb* is to the trapezium CD*ba* (or the triangle ABC) in the ſame proportion. In like manner, the ſum of the triangles APH, HMB, BNF, FVC is equal to the trapezium GR*dc*; becauſe the ſum of theſe triangles is to the ſum of the triangles AHB, BFC as P*l* (or H*q*) is to HL, or as the ſquare of P*q* is to the

ſquare

ſquare of AL, or as the ſquare of RG is to the ſquare of AG; that is, as one is to four: and the trapezium GR*dc* is to the trapezium D*Gcb* (which is equal to the ſum of the triangles AHB, BFC) in the ſame proportion. Thus it appears, that the ſum of the triangles which are ſubducted from the parabolic ſegments, and added to the inſcribed polygon at every new biſection of the parts of the baſe AC, is always equal to the trapezium that is at the ſame time added to CR*da* by biſecting AR the part of the baſe next adjoining to the point A; and therefore the polygon APHMBNFVC inſcribed in the parabolic ſegment is always equal to the trapezium CR*da* terminated by the given right line C*a* and a parallel R*d* drawn through R the diviſion of the baſe next adjoining to the point A. This trapezium CR*da* approaches continually to its limit the triangle CA*a*; the inſcribed polygon at the ſame time approaches to the parabolic ſegment its limit; and theſe are equal to each other, as we have demonſtrated. After all the methods that have been propoſed for demonſtrating the quadrature of the parabola, this, which we have deſcribed from the inventor, ſeems to have a particular elegance. On this occaſion, he ſhews how to find the ſum of any number of terms that decreaſe conſtantly in the proportion of four to one; and, by this example of a geometrical progreſſion, (as it is commonly called,) opened up a ſubject which has been treated at great length by the modern Geometricians.

ARCHIMEDES having demonſtrated the quadrature of the parabola; having alſo ſhewn how to approximate to the area of the circle and ellipſe, and how to compare ſpheres, ſpheroids and conoids, or any portions of theſe ſolids, with given cylinders or cones; there ſeems to have been nothing neglected by him that was neceſſary to complete the menſuration of all the figures then received into geometry, and of the ſolids generated by them, if it was not an approximation to the hyperbolic areas. But this does not appear to have been much conſidered by Geometricians, till the analogy of theſe areas with the logarithms was obſerved; and the ſolution of the moſt difficult problems of this kind was found to depend on the meaſures of angles or the areas

reas of the circle, and the meaſures of ratios or the areas of the hyperbola. Beſides what relates to the circle and conic ſections, and the ſolids generated by them, he has given us a treatiſe concerning the ſpiral line which is deſcribed by a point moving with a given velocity along a right line that revolves at the ſame time with an uniform angular motion about one of its extremities. The proportion of the area of this curve to the area of the circumſcribed circle is eaſily deduced from the principles which have been already demonſtrated.

FIG. 14. Let C be the beginning of the ſpiral, CE the ſituation of the revolving ray at the beginning of the motion; and the angles ECA, ACB, BCD, DCE being equal, let CA, CB, CD, CE meet the ſpiral in A, B, D and E: Then, the angular motion of the ray, and the motion of the point that deſcribes the curve along the ray, being both uniform, the right lines CA, CB, CD, CE muſt always increaſe by the ſame difference equal to the firſt line CA. From the center C deſcribe the ſimilar arches AG, BI, DL, EN, which meet CE, CA, CB, CD in G, I, L and N without the ſpiral, as alſo the ſimilar arches AH, BK, DM that are all within the ſpiral, and meet CB, CD, CE in H, K and M. The ſpiral area CABDE is always a limit betwixt the ſum of the circumſcribed ſectors CGA, CIB, CLD, CNE and the ſum of the inſcribed ſectors CAH, CBK, CDM. The circle deſcribed with the radius CE is the ſum of as many ſectors, equal to the greateſt CNE, as there are ſectors; and therefore (ſince the ſectors are as the ſquares of CA, CB, CD, CE) one third part of the area of this circle is alſo a limit betwixt the ſums of the ſame circumſcribed and inſcribed ſectors. The difference of the ſums of theſe ſectors is equal to CNE, which, by biſecting the angles at C, may become leſs than any ſpace that can be aſſigned: From which it follows, that the ſpiral area CABDE is equal to one third part of the circle of the radius CE. In like manner, the whole ſpiral area generated by the ray drawn from the point C to the curve while it makes two revolutions, is the third part of a ſpace that is double of the circle deſcribed with a radius equal to 2CE; and the whole area generated by the ray from the beginning of the motion till after any number of revolutions, is

equal

equal to the third part of a ſpace that is the ſame multiple of the circle deſcribed with the greateſt ray, as the number of revolutions is of unit. Any portion of the area of the ſpiral that is terminated by the curve C*m*A and the right line CA is ſhewn, in the ſame manner, to be equal to one third part of the ſector CAG, terminated by the right line CA, and CG the ſituation of the revolving ray when the point that deſcribes the curve ſets out from C. We ſhall afterwards give an account of his theorems concerning the tangents of this curve.

PAPPUS * takes occaſion, from the ſpiral of ARCHIMEDES, to conſider that which is deſcribed by ſimilar motions on the ſurface of a ſphere, and finds a portion of the ſurface terminated by this ſpiral to be equal to the ſquare of the diameter. Let C be the center of the ſphere, ARBA a great circle, P its pole, FIG. 15. and while the quadrant PMA revolves about the pole P with an uniform motion, let a point proceeding from P move with a given velocity along the quadrant, and trace upon the ſpherical ſurface the ſpiral PF*fa*. Let PMA be the ſituation of the quadrant at the beginning of the motion, and let the point that deſcribes the ſpiral come to *a* when the quadrant comes to the ſituation P*ma*. Suppoſe any two quadrants PFR, P*fr* to meet the ſpiral in F, *f* and the circle ARB in R and *r*; through F and *f* from the pole P deſcribe the circles MFN, L*f*Q, meeting the ſemicircle APB in M, N, L and Q; let the circle MFN meet P*fr* in H, and L*f*Q meet PFR in K, and H*f* ſhall be to R*r* as the quadrant PMA is to the arch A*a*, or as the velocity of the point, which deſcribes the ſpiral, along the quadrant PFR to the velocity of R in the circle ARB. Upon CP let C*p* be taken ſo that the ſquare of CP may be to the ſquare of C*p* in the ſame proportion. From C as center deſcribe the circle *pnb* in the ſame plane with APB; produce CN and CQ till they meet this circle in *n* and *q*; from the center *p* deſcribe through *n* and *q* the arches *nu*, *qs* meeting *pq*, *pn* in the points *u* and *s*. Then, ſince the area PFH is to PR*r* as the ſquare of the chord PF is to the ſquare of the chord PR, and the a-

* Collect. Mathem. lib. 4. prop. 30.

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rea PR*r* is to the ſector CR*r* as the ſquare of PR is to the ſquare of CR, the area PFH is to CR*r* as the ſquare of PF (or PN) is to the ſquare of CR, or CN. But CR*r* is to CNQ as R*r* is to NQ, and therefore as the ſquare of C*p* is to the ſquare of CP, or as the ſquare of *pn* is to the ſquare of PN; ſo that the area PFH is to the ſector CNQ as the ſquare of *pn* is to the ſquare of CN. The angle *npu* being equal to one half of the angle NCQ, the ſector *pnu* is to one half of the ſector CNQ in the ſame proportion of the ſquare of *pn* to the ſquare of CN: Therefore the area PFH is double of the ſector *pnu*; and, in the ſame manner, the area PK*f* is double of the ſector *pqs*. If we ſuppoſe the quadrants from the pole P to divide the arch A*a* into any number of equal parts, and to meet the ſpiral in any points as F, *f*, the right lines CN, CQ produced to the circle *pnb* will divide the quadrant *pnb* into the ſame number of equal parts; and the ſpaces PFH inſcribed in the area PF*fa*P being always double of the ſectors *pnu* inſcribed in the ſegment *pnbp*, the ſpaces PK*f* that are deſcribed about the ſpiral area being always double of the ſectors *pqs* deſcribed about that ſegment, and the difference betwixt the ſum of the circumſcribed and inſcribed ſectors being equal to the greateſt ſector, (which, by continuing to biſect the parts of the arch A*a*, or quadrant *pnb*, may become leſs than any aſſignable quantity;) therefore the area PF*a*P terminated by the ſpiral PF*a* and the right lines P*a*, P*ma* is double of the ſegment *pnbp*. The arch AR*a* is to the quadrant PMA (and the ſector CA*a* to the quadrant CPNB) as the ſquare of C*p* is to the ſquare of CP, or as the area C*pnb* to the area CPNB; and therefore the ſector CA*a* is equal to the quadrant C*pnb*. But it follows, from what has been demonſtrated above after ARCHIMEDES, that the ſurface PMAR*am*P is double of the ſector CA*a*; and therefore it is alſo double of the quadrant C*pnb*. From this ſurface take away the part PZF*am*P, and there remains the ſurface PMAR*a*FZP (bounded by the quadrant PMA, the arch AR*a* and the ſpiral PZF*a*) double of the triangle C*pb*, and therefore equal to the ſquare of the right line C*p*.

Suppoſe, for example, the quadrant PMA to make a compleat revolution in the ſame time that the point which traces the ſpiral

ral on the ſurface of the ſphere deſcribes the quadrant, which is the caſe conſidered by PAPPUS. Then, Rr being quadruple of Hf or NQ, Cp muſt be double of CP, and therefore equal to AB the diameter of the ſphere. The portion of the ſpherical ſurface terminated by the whole ſpiral, the circle ARBA and the quadrant PMA, is equal to the ſquare of AB. In any other caſe the area PMAαFαP is to the ſquare of the diameter AB, in the ſame proportion as the arch Aα is to the whole circumference ARBA; and this area is always to the ſpherical triangle PAα, as the inſcribed ſquare is to the circle.

It is demonſtrated in the ſame manner, that the area PZFP is double of the ſegment *pxnp*. But the ſector CAR is to the ſector CPN, as the arch AR is to the Arch PN, and conſequently, as the ſquare of Cp is to the ſquare of CP, or as the ſector *cpn* is to CPN; ſo that the ſector CAR is equal to *cpn*: and the ſpherical triangle PAR, being double of the ſector CAR, it is alſo double of *Cpn*. From this ſpherical triangle ſubduct the area PZFP, and the remainder PMARFZP muſt be double of the triangle *Cpn*. Therefore the portion of the ſpherical ſurface, terminated by the quadrant PMA, the arches AR, FR, and the ſpiral PZF, admits of a perfect quadrature, when the ratio of Cp to CP, or of the arch Aα to the whole circumference, can be aſſigned.

We have now given a ſummary account, of the progreſs that was made by the ancients in meaſuring and comparing curvilineal figures, and of the method by which they demonſtrated all their theorems of this kind. It is often ſaid, that curve lines have been conſidered by them as polygons of an infinite number of ſides. But this principle no where appears in their writings. We never find them reſolving any figure, or ſolid, into infinitely ſmall elements. On the contrary, they ſeem to avoid ſuch ſuppoſitions, as if they judged them unfit to be received into geometry, when it was obvious that their demonſtrations might have been ſometimes abridged by admitting them. They conſidered curvilineal areas as the limits of circumſcribed or inſcribed figures of a more ſimple kind, which approach to theſe

limits, (by a biſection of lines, or angles, that is continued at pleaſure,) ſo that the difference betwixt them may become leſs than any given quantity. The inſcribed or circumſcribed figures were always conceived to be of a magnitude and number that is aſſignable; and from what had been ſhewn of theſe figures, they demonſtrated the menſuration, or the proportions, of the curvilineal limits themſelves, by arguments *ab abſurdo*. They had made frequent uſe of demonſtrations of this kind from the beginning of the Elements; and theſe are in a particular manner adapted for making a tranſition from right-lined figures to ſuch as are bounded by curve lines. By admitting them only, they eſtabliſhed the more difficult and ſublime part of their geometry on the ſame foundation as the firſt elements of the ſcience. Nor could they have propoſed to themſelves a more perfect model.

We have already obſerved, how ſolicitous ARCHIMEDES appears to be, that his demonſtrations ſhould be found to depend on thoſe principles only that had been univerſally received before his time. In his treatiſe of the quadrature of the parabola, he treats of a progreſſion whoſe terms decreaſe conſtantly in the proportion of four to one, which we expreſſed by the trapezia
FIG. 13. CD*ba*, DG*cb*, GR*dc*, &c. But he does not ſuppoſe this progreſſion to be continued to infinity, or mention the ſum of an infinite number of terms; though it is manifeſt, that all which can be underſtood by thoſe who aſſign that ſum was fully known to him. He appears to have been more fond of preſerving to the ſcience all its accuracy and evidence, than of advancing paradoxes; and contents himſelf with demonſtrating this plain property of ſuch a progreſſion, That the ſum of the terms continued at pleaſure, added to the third part of the laſt term, amounts always to four thirds of the firſt term; as the ſum of the trapezia CD*ba*, DG*cb*, GR*dc*, added to one third part of the laſt trapezium GR*dc*, amounts always to four thirds of the firſt trapezium CD*ba*, or to the triangle CA*a*. Nor does he ſuppoſe the chords of the curve to be biſected to infinity; ſo that after an infinite biſection the inſcribed polygon might be ſaid to coincide with the parabola. Theſe ſuppoſitions had been

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new to the Geometricians in his time; and ſuch he appears to have carefully avoided.

He has demonſtrated many other theorems of this kind from the properties of certain progreſſions, the terms of which correſpond to the circumſcribed and inſcribed figures: but he never ſuppoſed theſe terms, or figures, to increaſe or decreaſe by infinitely ſmall differences, and to become infinite in number; that their ſum might be ſuppoſed equal to the curvilineal area, or ſolid. It was ſufficient for his purpoſe, to aſſign a quantity that is always a limit betwixt the ſum of all the terms of the progreſſion, and the ſame ſum without one of the extreme terms; as the area, or ſolid, is always a limit between the ſum of the circumſcribed and the ſum of the inſcribed figures, which ſums differ from each other in the extreme figures only. He conſidered but one decreaſing geometrical progreſſion, and ſhewed how to find in an arithmetical progreſſion the ſum of the terms and of their ſquares only. Of late, other geometrical progreſſions have been employed ſucceſsfully, for meaſuring the areas of curves; the ſums of the cubes, and of all the other powers, of the terms in an arithmetical progreſſion have ſerved for the ſame purpoſe: and each of his diſcoveries has produced ſome extenſive theory in the modern geometry.

His method has been often repreſented as very perplexed, and ſometimes as hardly intelligible. But this is not a juſt character of his writings, and the ancients had a different opinion of them *. He finds it neceſſary indeed to premiſe ſeveral propoſitions to the demonſtration of the principal theorems; and on this account his method has been excepted againſt as tedious. But the number of ſteps is not the greateſt fault a demonſtration may have; nor is this number to be always computed from thoſe that may be propoſed in it, but from thoſe that are neceſ-

* Plutarch celebrates the ſimplicity and plainneſs with which he treats the moſt difficult and abſtruſe queſtions: Οὐ γὰρ ἐστιν ἐν γεωμετρίᾳ καλεπωτέρας καὶ βαρυτέρας ἀποδείσεις ἐν ἁπλουστέροις λαβεῖν καὶ καθαρωτέροις στοιχείοις γραφομένας. *Plut. in vita Marcelli.*

ſary to make it full and concluſive. Beſides, theſe preliminary propoſitions are generally valuable on their own account, and render our view of the whole ſubject more clear and compleat. In his treatiſe of the ſphere and cylinder, for example, by his demonſtrating ſo fully the menſuration of the ſurfaces and ſolids, generated by the internal and external polygons, we not only ſee how the ſurface and ſolid content of the ſphere itſelf is determined, but we acquire a more perfect knowledge of this theory, and of all that relates to it, with a ſatisfaction that we are ſenſible is often wanting in the incompleat demonſtrations of ſome other methods.

By ſo many valuable diſcoveries demonſtrated in ſo accurate a manner, and by the admirable uſe he made of his knowledge in the celebrated ſiege of his native city *, and upon other occaſions †, ARCHIMEDES has diſtinguiſhed himſelf amongſt the Geometricians, and has done the greateſt honour to this part of learning. He has not however eſcaped the cenſures of ſome writers, who being unskilful in geometry, and unable to reconcile their own conceits with his demonſtrations, have repreſented him as in an error, and miſleading Mathematicians by his authority ||. But though Mathematicians may be grateful, authority has not any place in this ſcience; and no Geometrician ever pretended, from the higheſt veneration for ARCHIMEDES, Sir ISAAC NEWTON or others, to reſt on their judg-

* How he diſconcerted all the efforts of two Roman armies, commanded by the Proconſul Marcellus and by Appius Claudius, in the ſiege of Syracuſe, (till the city being taken by ſurpriſe and treachery, an end was put to his life and enquiries at once,) is deſcribed at length by Polybius, Livy, Plutarch, *&c.* He was called πολύμηχανος and ἑκατόγχειρ; and, according to Plutarch, acquired the reputation of more than human learning. Medals of Syracuſe, with figures that are ſuppoſed to refer to his diſcoveries, ſerve rather to juſtify his countrymen from the reproach of ingratitude which ſome have imputed to them, than to do honour to the immortal Archimedes. *Paruta Sicil. Spanhem. in orat. 1. Juliani.*

† Diodorus Siculus tells us, (*lib.* 5.) that when Archimedes travelled into Egypt, he invented machines that were of great uſe to that nation, and procured him an univerſal reputation.

‖ Decepit illos auctoritas Archimedis, *&c.* *Hobbes de principiis & ratiocinatione Geometrarum.* The learned Joſeph Scaliger and others have alſo writ againſt him.

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ment in a matter of geometrical demonſtration. The purſuit of general and eaſy methods may have induced ſome to make uſe of exceptionable principles; and the vaſt extent, which the ſcience has of late acquired, may have occaſioned their propoſing incompleat demonſtrations. They may have alſo ſometimes fallen into miſtakes: but it will be found difficult to aſſign one falſe propoſition that has been ever generally received by Geometricians; and it is hardly poſſible, that accuſations of this nature can be more miſplaced.

In what ARCHIMEDES had demonſtrated of the limits of figures and progreſſions, there were valuable hints towards a general method of conſidering curvilineal figures; ſo as to ſubject them to menſuration by an exact quadrature, an approximation, or by comparing them with others of a more ſimple kind. Such methods have been propoſed of late in various forms, and upon different principles. The firſt eſſays were deduced from a careful attention to his ſteps *. But, that his method might be more eaſily extended, its old foundation was abandoned, and ſuppoſitions were propoſed which he had avoided. It was thought unneceſſary to conceive the figures circumſcribed or inſcribed in the curvilineal area, or ſolid, as being always aſſignable and finite; and the precautions of ARCHIMEDES came to be conſidered as a check upon Geometricians, that ſerved only to retard their progreſs. Therefore, inſtead of his aſſignable finite figures, indiviſible or infinitely ſmall elements were ſubſtituted; and theſe being imagined indefinite, or infinite, in number, their ſum was ſuppoſed to coincide with the curvilineal area, or ſolid.

It was however with caution that theſe ſuppoſitions were at firſt employed in geometry by CAVALERIUS, the ingenious author of the method of indiviſibles, and by others. He diſcovered a method, which he found to be of a very extenſive uſe,

* C'etoit en obſervant de prês la marche d'Archimede qu'il [*M. de Roberval*] etoit arrivé à cette ſublime & merveilleuſe ſcience, *&c. Ouvrag. de l' Acad. Royal.* 1693. This is generally acknowledged by the writers of that time.

and

and of an eaſy application, for meaſuring or comparing planes and ſolids; and would not deprive the public of ſo valuable an invention. In propoſing it, he ſtrove to avoid * the ſuppoſing magnitude to conſiſt of indiviſible parts, and to abſtract from the contemplation of infinity; but he acknowledged, that there remained ſome difficulties in this matter which he was not able to reſolve. Therefore he ſubjoined more unexceptionable demonſtrations to thoſe he had deduced from his own principles; and the diſputes which enſued (the firſt of any moment that were known between Geometricians) juſtified his precautions. Afterwards, infinitely ſmall elements were ſubſtituted in place of his indiviſibles; and various improvements were made in this doctrine. The method of ARCHIMEDES, however, was often kept in view, and frequently appealed to as the ſureſt teſt of every new invention. The harmony betwixt the concluſions that aroſe from the old and new methods contributed not a little to the credit which the latter at firſt acquired; till being more and more reliſhed, they came at length to be generally admitted on their own evidence, and ſeem'd to merit ſo favourable a reception, by the great advantages that were derived from them for reſolving the moſt difficult problems, and demonſtrating the moſt general theories, in a brief and eaſy manner.

But when the principles and ſtrict method of the ancients, which had hitherto preſerved the evidence of this ſcience entire, were ſo far abandoned, it was difficult for the Geometricians to determine where they ſhould ſtop. After they had indulged themſelves in admitting quantities, of various kinds, that were not aſſignable, in ſuppoſing ſuch things to be done as could not poſſibly be effected, (againſt the conſtant practice of the ancients,) and had involved themſelves in the mazes of infinity; it was not eaſy for them to avoid perplexity, and ſometimes error,

* Quoad continui compoſitionem, manifeſtum eſt ex præoſtenſis, ad ipſum ex indiviſibilibus componendum nos minimè cogi: ſolùm enim continua ſequi indiviſibilium proportionem, & è converſo, probare intentum fuit; quod quidem cum utraque poſitione ſtare poteſt. Tandem verò dicta indiviſibilium aggregata non ita pertractavimus, ut infinitatis rationem propter infinitas lineas ſeu plana ſubire videantur, *&c. Cavalerii Geom. indiviſ. lib. 7. praf.*

or

or to fix bounds to theſe liberties when they were once introduced. Curves were not only conſidered as polygons of an infinite number of infinitely little ſides, and their differences deduced from the different angles that were ſuppoſed to be formed by theſe ſides; but infinites and infiniteſimals were admitted of infinite orders, every operation in geometry and arithmetic applied to them with the ſame freedom as to finite real quantities, and ſuppoſitions of this nature multipled, till the higher parts of geometry (as they were moſt commonly deſcribed) appeared full of myſteries.

From geometry the infinites and infiniteſimals paſſed into philoſophy, carrying with them the obſcurity and perplexity that cannot fail to accompany them. An actual diviſion, as well as a diviſibility of matter *in infinitum*, is admitted by ſome. Fluids are imagined conſiſting of infinitely ſmall particles, which are compoſed themſelves of others infinitely leſs; and this ſubdiviſion is ſuppoſed to be continued without end. Vortices are propoſed, for ſolving the phænomena of nature, of indefinite or infinite degrees, in imitation of the infiniteſimals in geometry; that, when any higher order is found inſufficient for this purpoſe, or attended with an inſuperable difficulty, a lower order may preſerve ſo favourite a ſcheme. Nature is confined in her operations to act by infinitely ſmall ſteps. Bodies of a perfect hardneſs are rejected, and the old doctrine of atoms treated as imaginary, becauſe in their actions and colliſions they might paſs at once from motion to reſt, or from reſt to motion, in violation of this law. Thus the doctrine of infinites is interwoven with our ſpeculations in geometry and nature. Suppoſitions, that were propoſed at firſt diffidently, as of uſe for diſcovering new theorems in this ſcience with the greater facility, and were ſuffered only on that account, have been indulged, till it has become crowded with objects of an abſtruſe nature, which tend to perplex it and the other ſciences that have a dependence upon it.

They who have made uſe of infinites and infiniteſimals with the greateſt liberty, have not agreed as to the truth and reality they would aſcribe to them. The celebrated Mr. LEIBNITZ owns

owns them to be no more than fictions. Others place them on a level with finite quantities, and endeavour to demonſtrate their reality, from magnitude's being ſuſceptible of augmentation and diminution without end, from the properties of the progreſſions of numbers that may be continued at pleaſure, and from the infinity which ſome Geometricians have aſcribed to the hyperbolic area. But in theſe arguments they ſeem to ſuppoſe the infinity which they would demonſtrate.

It was a principle of the ancient Geometricians, That any given line may be produced, and its parts ſubdivided, at pleaſure: but they never ſuppoſed it to be produced, till it ſhould become infinitely great; or to be ſubdivided, till its parts ſhould become infinitely ſmall. It does not neceſſarily follow, that, becauſe any given right line may be continued further, it can be produced till it become actually infinite, or that we are able to conceive ſuch a line to be deſcribed, ſo as to admit it in geometry. In general, magnitude is capable of being increaſed without end; that is, no term or limit can be aſſigned or ſuppoſed beyond which it may not be conceived to be further increaſed. But from this it cannot be inferred, that we are able to conceive or ſuppoſe magnitude to be really infinite *: or, if we are

* In a late treatiſe aſcribed to a celebrated author, juſtly eſteemed for his various writings, ſeveral arguments are propoſed, for admitting magnitude actually infinite: not that kind which has no limits, comprehends all, and can receive no addition, which he calls *metaphyſical*; but that which he defines to be greater than any finite magnitude, which he diſtinguiſhes from the former, and calls *geometrical*. *Puiſque la grandeur eſt ſuſceptible d'augmentation ſans fin on la peut concevoir ou ſuppoſer augmentée une infinité des fois, c'eſt-à dire qu'elle ſera devenuë infinie. Et, en effet, il eſt impoſſible que la grandeur ſuſceptible d'augmentation ſans fin ſoit dans le même cas que ſi elle n'en etoit pas ſuſceptible ſans fin. Or, ſi elle ne l'etoit pas, elle demeureroit toujours finie; donc etant ſuſceptible d'augmentation ſans fin, elle peut ne demeurer pas toujours finie, ou, ce qui eſt la même, devenir infinie.* Elem. de la geom. de l'infini, § 83. Becauſe magnitude is ſuſceptible of augmentation without end, the author concludes, that we may ſuppoſe it augmented an infinite number of times. But, by being ſuſceptible of augmentation without end, we underſtand only, that no magnitude can be aſſigned or conceived ſo great but it may be ſuppoſed to receive further augmentation, and that a greater than it may ſtill be aſſigned or conceived. We eaſily conceive that a finite magnitude may become greater and greater without end, or that no termination

are able to join infinity to any ſuppoſed idea of a determinate quantity, and to reaſon concerning magnitude actually infinite, it is not ſurely with that perſpicuity that is required in geometry.

mination or limit can be aſſigned of the increaſe which it may admit: but we do not therefore clearly conceive magnitude increaſed an infinite number of times. Mr. Lock acknowledges, that we eaſily form an idea of the infinity of number, to the end of whoſe addition there is no approach: but he diſtinguiſhes betwixt this and the idea of an infinite number; and ſubjoins, that how clear ſoever our idea of the infinity of number may be, there is nothing more evident than the abſurdity of the actual idea of an infinite number.

The latter part of the argument amounts to this: "It is impoſſible that magnitude being ſuſceptible of augmentation without end, can be in the ſame caſe as if it was not ſuſceptible of augmentation without end. But if it was not ſuſceptible of augmentation without end, it would remain always finite. Therefore, ſince it is ſuſceptible of augmentation without end, the contrary muſt be allowed; that is, it may not always remain finite, or it may become infinite." The Force of which argument ſeems to be taken off, by conſidering, that, if magnitude was not ſuſceptible of augmentation without end, it would not only remain always finite, but there would neceſſarily be a term, limit or degree of magnitude which could never be exceeded, or there might be a greateſt magnitude. And, by allowing that there is no ſuch term or limit, magnitude is not ſuppoſed to be in the ſame caſe as if it was not ſuſceptible of augmentation without end, though we ſhould refuſe that it may become infinite. What is oppoſite to the ſuppoſing magnitude ſuſceptible of augmentation without end, is not the ſuppoſing it always finite, (for finite magnitude is capable of being increaſed without end;) but the ſuppoſing it ſuſceptible of no augmentation at all, or of an augmentation that has a limit or end.

The ſeries of numbers, 1, 2, 3, 4, &c. in their natural order, may be continued without end; and it is ſaid, that "we never come nearer the end of the progreſſion, how great ſoever the number may be to which we arrive; which is a character that cannot belong to a ſeries of a finite number of terms. Therefore this natural ſeries has an infinite number of terms." And it is added, that "though we can go over a finite number of terms only, yet all the terms of this infinite progreſſion are equally real." But if we can conceive this ſeries to have any end, it ſeems to be evident, that we muſt approach to this end as we proceed from the beginning towards it; and that, while we advance, the diſtance of any term from the end muſt decreaſe (whether this diſtance be called finite or infinite) by the ſame quantity as the diſtance from any ſubſequent term decreaſes, or the diſtance from the beginning of the ſeries increaſes. If we cannot conceive the ſeries to have an end, then we can have no idea of its laſt term. If we ſuppoſe this ſeries to be continued to infinity, it would indeed be abſurd, after ſuch a ſuppoſition, to ſay that the number of its terms is finite. But, in treating this ſcience ſtrictly, it may perhaps be better to avoid this ſuppoſition. For if it is only a finite number of terms we can clearly conceive, how ſhall we judge of the reality of the reſt? or wherein ſhall we place the reality of thoſe which it is impoſſible for us to aſſign? of which two kinds are ſaid to be in this ſame ſeries,

try. In the ſame manner, no magnitude can be conceived ſo ſmall, but a leſs than it may be ſuppoſed; but we are not therefore able to conceive a quantity infinitely ſmall. A given magnitude

each infinite in number; the firſt of which are ſaid to be finite, but indeterminable; the latter, actually infinite.

The argument from the infinity of the hyperbolic area is much inſiſted on. "The hyperbolic area (*Elem. de la geom. de l'infin. pref.*) is as really infinite, as "a determined parabolic area is two thirds of the circumſcribed parallelogram. "It is trifling to ſay, that the one can be actually deſcribed, and the other can- "not. Geometry is entirely intellectual, and independent of the actual deſcri- "ption and exiſtence of the figures whoſe properties it diſcovers. All that is "conceived neceſſary in it has the reality which it ſuppoſes in its object. There- "fore the infinite which it demonſtrates is as real as that which is finite, *&c.*" And the learned author, after inſiſting on this ſubject, concludes, that, "not to "receive infinity as it is here repreſented, with all its neceſſary conſequences, is "to reject a geometrical demonſtration; and that he who rejects one, ought to "reject them all." But though the actual deſcription of the figures which are conſidered in geometry be not neceſſary, yet it is requiſite that we ſhould be able clearly to conceive that they may exiſt; and a diſtinct idea of the manner how they may be ſuppoſed to be deſcribed or generated is neceſſary, that they may have a place in this ſcience. Principles that are propoſed as of the moſt extenſive uſe, and as the foundation of all the ſublime geometry, ought to be clear and unexceptionable. If this ſcience is entirely intellectual, or if the reality of its objects is to be conſidered as having a dependence on their being conceived by the mind, it would ſeem that there muſt be a difference betwixt the reality of finite aſſignable lines or numbers, and the reality we can aſcribe to infinite lines or numbers, which are not aſſignable, and cannot be ſuppoſed to be produced or generated but in a manner that is allowed to be inconceivable. As for what is ſaid of the parabolic and hyperbolic areas, we can conceive any portion of the parabola to be accurately deſcribed, and its area to be determined, though no exact figure of this kind ſhould ever exiſt. We can alſo conceive, that the hyperbola and its aſymptote may be produced to any aſſignable diſtance: but we do not ſo clearly conceive that they may be produced to a diſtance greater than what is aſſignable; and we may well be allowed to heſitate at ſuch a ſuppoſition in ſtrict geometry. Any finite ſpace being propoſed, the hyperbolic area (terminated by the curve, the aſymptote and a given ordinate) will exceed it by producing the curve and aſymptote to an aſſignable diſtance; and there is no aſſignable limit in this (as in ſome other caſes) which the area may not ſurpaſs in magnitude. Therefore it is ſaid, that this area would be infinite, if the curve and aſymptote could be infinitely produced. But no argument for admitting magnitude actually infinite can be deduced from this, which does not more eaſily appear from hence, that a parallelogram of a given height would be infinite if it could have an infinite baſe: from which it cannot be inferred that ſuch a baſe or parallelogram can actually exiſt. It is often ſaid, that a rectangle of a given height on an imaginary baſe (as the Analyſts ſpeak) is imaginary: but we cannot thence infer, that an imaginary line or rectangle can exiſt. It is not however,

gnitude may be ſuppoſed to be divided into any aſſignable number of parts; but it cannot therefore be conceived to be divided into a number of parts greater than what is aſſignable. The parts

ever our intention to maintain the impoſſibility of infinite magnitude; but to ſhew, that ſuch doctrines are not neceſſary conſequences of the received principles of this ſcience, and not very proper to be admitted as the ground-work of the high geometry.

As for the hyperbolic areas of a higher kind, which are ſaid to be of a finite magnitude though infinitely produced, the meaning is, that there is a certain finite ſpace which ſuch an area never can equal, though the curve and its aſymptote be produced never ſo far; to which however the area approaches, ſo that the exceſs of that finite ſpace above it may become leſs than any ſpace that may be propoſed, by producing the curve and its aſymptote to a diſtance that is aſſignable: As, the ſum of the trapezia CD*ba*, DG*cb*, GR*dc*, *&c.* that are determined by biſecting AR continually, is always leſs than the triangle AC*a*, but approaches to it ſo that their difference AR*d* may become leſs than any given ſpace O: Or, as the ſum of the right lines CD, DG, GR, *&c.* is always leſs than CA, but approaches to it, ſo that, by continuing the biſection, the difference AR may become leſs than any aſſigned quantity. But we ſhall have occaſion to treat of theſe afterwards more fully. FIG. 13.

In the ſame treatiſe (§ 196.) a proof is offered, to ſhew, that, in the infinite ſeries of numbers proceeding in their natural order, there are finite numbers whoſe ſquares become infinite, which are called indeterminable, and are ſuppoſed to occupy the obſcure paſſage from the numbers that are aſſignable to thoſe that are infinite. A greateſt finite ſquare is ſuppoſed in this progreſſion, and repreſented by *nn*; all that preceed it are finite, and all that follow after it are ſuppoſed infinite. The numbers in this progreſſion between *n* and *nn*, being leſs than *nn*, are finite; but being greater than *n*, their ſquares are greater than *nn*, and therefore, by the ſuppoſition, are infinite. But how can we admit the ſuppoſition of a greateſt finite ſquare number, ſuch as is here expreſſed by *nn*? The number *nn*, being finite, is not the next to it in the progreſſion, (which exceeds it by unit only,) alſo finite. Should we allow, that a finite number becomes infinite by adding unit to it, or even by ſquaring it, how ſhall we diſtinguiſh finite from infinite? We commonly conceive finite magnitude to be aſſignable, or to be limited by ſuch as are aſſignable, and to be ſuſceptible of further augmentation: and therefore infinite magnitude would ſeem to imply, either that which exceeds all aſſignable magnitude, or that which cannot admit of any further augmentation; theſe being directly oppoſite to what we moſt clearly conceive of finite magnitude. But neither of theſe conſtitute the idea of infinite magnitude, as it muſt be underſtood in that treatiſe. The former is applicable to thoſe numbers which the author calls finite and indeterminable; which, being ſuppoſed to produce infinite ſquares, muſt therefore exceed all aſſignable numbers whoſe ſquares are aſſignable and finite. The latter is aſcribed to that infinite only which he calls metaphyſical, and excludes from geometry. We are at a loſs to form a diſtinct idea even of finite itſelf as it is here underſtood; and it would ſeem,

parts of a given line may be ſuppoſed to be continually biſected till they become leſs than any line that is propoſed; and this is ſufficient for completing the demonſtrations of the ancients.

But that the more art and ingenuity is employed in penetrating into the doctrine of infinites, it becomes the more abſtruſe.

A proof is offered *à poſteriori* (§ 393.) to ſhew, that there are finite fractions in the ſeries $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, *&c.* whoſe ſquares become infinitely little in the ſeries $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}$, *&c.* The ſum of the firſt ſeries correſponds with the area included betwixt the common hyperbola and its aſymptote; and is ſaid to be infinite when the ſeries is ſuppoſed to be continued to infinity. The ſum of the latter ſeries correſponds with the area of an hyperbola of a higher order, and is ſaid to be finite, even when the ſeries is ſuppoſed to be continued to infinity; becauſe there is a limit which this ſum can never equal, to which however it continually approaches, as we have already deſcribed. This being allowed, it is ſuppoſed farther, that there is an infinite number of finite terms in the firſt progreſſion; and it is thence demonſtrated, that there are finite fractions in the firſt ſeries whoſe ſquares become infinitely little in the ſecond, thus: "If it ſhould be pretended, "that all the finite terms in the firſt ſeries have their ſquares finite in the ſecond, "there would be an infinite number of finite terms in the ſecond as well as in "the firſt; and the ſums of both would be infinite: ſo that the contrary of an "undoubted truth, that is univerſally received, would be demonſtrated." If we could allow that there is an infinite number of finite terms in the firſt ſeries, this argument might have ſome weight. But this is a ſuppoſition we cannot admit. For the denominator of any fraction in the firſt ſeries is always equal to the number of terms from the beginning, and muſt be ſuppoſed infinite when the number of terms is ſuppoſed infinite; but a fraction that has unit for its numerator, and is ſuppoſed to have an infinite number for its denominator, cannot be ſuppoſed finite, but infinitely little: ſo that we cannot ſuppoſe an infinite number of terms in the firſt ſeries to be finite. It is often ſaid in this treatiſe, that there is an infinite number of finite terms in the natural ſeries 1, 2, 3, 4, 5, *&c.* continued to infinity. But we are at a loſs to conceive how this can be admitted; ſince, in any ſuch progreſſion, the laſt or greateſt term is always equal to the number of terms from the beginning, and cannot be ſuppoſed finite when the number of terms is ſuppoſed infinite. There is an aſſignable limit which the ſum of the terms of the ſecond ſeries never amounts to; but there is no aſſignable limit which the ſum of the firſt ſeries may not ſurpaſs, (as we ſhall ſhew afterwards;) and the ſum of the terms of the firſt is greater than the ſum of the correſponding terms of the ſecond, in a ratio that by continuing the terms may exceed any aſſignable ratio of a greater magnitude to a leſſer: and as this is eaſily underſtood and demonſtrated, ſo there is no neceſſity for having recourſe to ſuch abſtruſe principles in order to account for it.

It is of no uſe to cite authorities on this ſubject, but as they may juſtify us in eſtabliſhing ſo noble a part of geometry for avoiding principles that are ſo much conteſted. What Ariſtotle taught of infinite magnitude is well known. Mr. Leibnitz, who for obvious reaſons cannot be ſuſpected of any prejudice againſt the doctrine of infinites, expreſſes himſelf thus: *On s'embaraſſe dans les ſeries des*

But it is acknowledged by thoſe who have treated the doctrine of infinites in the fulleſt manner, that "there is ſomething in- "conceivable in ſuppoſing an infinitely great or infinitely ſmall "number or figure to be produced or generated; and that the "paſſage from finite to infinite is obſcure and incomprehen- "ſible:" and therefore it is better for us, in treating of ſo ſtrict a ſcience as geometry, to abſtract from theſe ſuppoſitions. The abſtruſe conſequences, that have been deduced from them by ingenious men, may the rather induce us to beware of admitting them as neceſſary principles in this ſcience, and to adhere to its ancient principles.

Mr. Lock, who wrote his excellent eſſay, "that we might "diſcover how far the powers of the underſtanding reach, to

des nombres qui vont à l'infini. On conçoit un dernier terme, un nombre infini, ou infinimens petit; mais tout cela ne ſont que des fictions. Tout nombre eſt fini & aſſignable, toute ligne l'eſt de même. Eſſai de Theodicée, diſc. prelim. § 70.

We have ſubjoined theſe remarks, at the deſire of ſome perſons for whom we have a great regard, to ſhew why we have not followed an author who has merited ſo well of Mathematicians, and who on every other occaſion has been juſtly applauded for his clear and diſtinct way of explaining the abſtruſe geometry. They who treated of infinites before him proceeded, as he obſerves, with a timorouſneſs which the contemplation of ſuch an object naturally inſpires: *Quand on y etoit arrivé,* (ſays he) *on s'arreſtoit avec une eſpece d'effroy & de ſainte horreur — on regardoit l'infini comme un miſtere qu'il falloit reſpecter, & qu'il n'etoit pas permis d'approfondir.* They ſtopt when they came to infinity with a ſort of holy dread, and reſpected it as an incomprehenſible myſtery. He adventures farther, in order to diſcover the ſource, and penetrate into the firſt principles of geometrical truth. Infinity, according to him, is the great trunk from which its various branches are derived, and to which they all lead. In this great purſuit he diſplays infinite and finite with a freedom that puts us in mind of the ancient Poet and his Gods, whom he repreſents with the paſſions of men, and mingles in their battles. We doubt not, that if a full and perfect account of all that is moſt profound in the high geometry could have been deduced from the doctrine of infinites, it might have been expected from this author: But our ideas of infinites are too obſcure and unadequate to anſwer this end; and there are many things advanced by all thoſe who have applied them with great freedom in geometry, that give ground to a remark like to Mr. de St. Evremond's, when he obſerves, that "it is ſurpriſing to find the ancient Poets ſo ſcrupulous to pre- "ſerve probability in actions purely human, and ſo ready to violate it in repre- "ſenting the actions of the Gods." Some have not only admitted infinites and infiniteſimals of infinite orders, but have diſtinguiſhed even nothings into various kinds: and if ſuch liberties continue, it is not eaſy to foreſee what abſurdities may be advanced as diſcoveries in what is called the ſublime geometry.

"what

" what things they are in any degree proportionate, and where " they fail us," obſerves, " that whilſt men talk and diſpute of " infinite magnitudes, as if they had as compleat and poſitive " ideas of them as they have of the names they uſe for them, " or as they have of a yard, or an hour, or any other determi- " nate quantity, it is no wonder if the incomprehenſible nature " of the thing they diſcourſe of, or reaſon about, leads them " into perplexities and contradictions; and their minds be over- " laid by an object too large and mighty to be ſurveyed and ma- " naged by them." Mathematicians indeed abridge their computations by the ſuppoſition of infinites; but when they pretend to treat them on a level with finite quantities, they are ſometimes led into ſuch doctrines as verify the obſervation of this judicious author. To mention an inſtance or two: The progreſſion of the numbers 1, 2, 3, 4, 5, *&c.* in their natural order, is ſuppoſed to be continued to infinity, till by the continual addition of units an infinite number is produced, which is conceived to be the termination of this ſeries. This infinite number is ſuppoſed to be ſtill capable of augmentation and diminution; and yet it is ſaid, " that it is neither increaſed nor diminiſhed " by the addition or ſubtraction of the ſame units from which " it was ſuppoſed to be generated." In a progreſſion of this kind, the number of terms is always equal to the laſt or greateſt term, and is finite when the laſt term is finite. If the number of terms be ſuppoſed infinite, the laſt term cannot be finite; and yet it is ſaid, " that in ſuch a progreſſion continued to in- " finity there is an infinite number of finite terms." It is evident, that no finite number can become infinite by the addition of unit or of any other finite number; and yet " a greateſt fi- " nite ſquare number is ſuppoſed in ſuch a progreſſion, the next " to which (though it exceed that finite number by an unit on- " ly) is ſuppoſed infinite." From theſe ſuppoſitions it is inferred, " that in ſuch a progreſſion continued to infinity there are " finite numbers whoſe ſquares become infinite;" though it ſeems very evident, that a finite number taken any finite number of times can never produce more than a finite number. We may perceive from theſe inſtances, that it is not by founding the higher geometry on the doctrine of infinites we can propoſe to

avoid

avoid the apparent inconſiſtencies that have been objected to it; and ſince an excellent author, who has always diſtinguiſhed himſelf as a clear and acute writer, has had no better ſucceſs in eſtabliſhing it on theſe principles, it is better for us to avoid them. Theſe ſuppoſitions however may be of uſe, when employed with caution, for abridging computations in the inveſtigation of theorems, or even for proving them where a ſcrupulous exactneſs is not required; and we would not be underſtood to affirm, that the methods of indiviſibles and infiniteſimals, by which ſo many unconteſted truths have been diſcovered, are without a foundation. We acknowledge further, that there is ſomething marvellous in the doctrine of infinites, that is apt to pleaſe and tranſport us; and that the method of infiniteſimals has been proſecuted of late with an acuteneſs and ſubtlety not to be parallelled in any other ſcience. But geometry is beſt eſtabliſhed on clear and plain principles; and theſe ſpeculations are ever obnoxious to ſome difficulties. If the greateſt accuracy has been always required in this ſcience, in reaſoning concerning finite quantities, we apprehend that Geometricians cannot be too ſcrupulous in admitting or treating of infinites, of which our ideas are ſo imperfect. Philoſophy probably will always have its myſteries. But theſe are to be avoided in geometry: and we ought to guard againſt abating from its ſtrictneſs and evidence the rather, that an abſurd philoſophy is the natural product of a vitiated geometry.

It is juſt at the ſame time to acknowledge, that they who firſt carried geometry beyond its ancient limits, and they who have ſince enlarged it, have done great ſervice, by deſcribing plainly the methods which they found ſo advantageous for this purpoſe, (though they might appear exceptionable in ſome reſpects,) that others might proceed with the ſame facility to improve it. Some of them have been ſo cautious as to verify their diſcoveries by demonſtrations in the ſtricteſt form; and others were able to have done this, had they not choſe rather to employ their time in extending the ſcience. At firſt, the variation from the ancient method was not ſo conſiderable, but that it was eaſy to have recourſe to it, when it ſhould be thought neceſſary

for

for the ſatisfaction of ſuch as required a ſcrupulous exactneſs. The Geometricians in the mean time made great improvements. They had the accurate method and examples of ARCHIMEDES before them, by which they might try their diſcoveries. Theſe ſerved to keep them from error, and the new methods facilitated their progreſs. Thus their views enlarged; and problems, that appeared at firſt ſight of an inſuperable difficulty, were afterwards reſolved, and came at length to be deſpiſed as too ſimple and eaſy. The menſuration of parabolas, hyperbolas, ſpirals of all the higher orders, and of the famous cycloid, were amongſt the earlieſt productions of this period; ſome of which ſeem to have been diſcovered by ſeveral Geometricians almoſt at the ſame time. It is not neceſſary for our purpoſe to deſcribe more particularly what diſcoveries were made by TORRICELLI, Meſſ. de FERMAT and de ROBERVAL, GREGORY à Sto. Vincentio, *&c.* by whom the theorems of ARCHIMEDES were continued, and applied to the menſuration of various figures.

The *Arithmetica infinitorum* of Dr. WALLIS was the fulleſt treatiſe of this kind that appeared before the invention of the method of fluxions. ARCHIMEDES had conſidered the ſums of the terms in an arithmetical progreſſion, and of their ſquares only, (or rather the limits of theſe ſums, deſcribed above,) theſe being ſufficient for the menſuration of the figures he had examined. Dr. WALLIS treats this ſubject in a very general manner, and aſſigns like limits for the ſums of any powers of the terms, whether the exponents be integers or fractions, poſitive or negative. Having diſcovered one general theorem that includes all of this kind, he then compounded new progreſſions from various aggregates of theſe terms, and enquired into the ſums of the powers of theſe terms, by which he was enabled to meaſure accurately, or by approximation, the areas of figures without number. But he compoſed this treatiſe (as he tells us) before he had examined the writings of ARCHIMEDES, and he propoſes his theorems and demonſtrations in a leſs accurate form. He ſuppoſes the progreſſions to be continued to infinity, and inveſtigates, by a kind of induction, the proportion of the ſum of the powers to the product that would ariſe by taking the greateſt

greateſt power as often as there are terms. His demonſtrations, and ſome of his expreſſions (as when he ſpeaks of quantities more than infinite) have been excepted againſt. But it was not very difficult to demonſtrate the greateſt part of his propoſitions in a ſtricter method; and this was effected afterwards by himſelf and others in various inſtances. He choſe to deſcribe plainly a method which he had found very commodious for diſcovering new theorems; and it muſt be owned, that this valuable treatiſe contributed to produce the great improvements which ſoon followed after. A like apology may be made for others who have promoted this doctrine ſince his time, but have not given us rigid demonſtrations. In general, it muſt be owned, that if the late diſcoveries were deduced at length, in the very ſame method in which the ancients demonſtrated their theorems, the life of man could hardly be ſufficient for conſidering them all: ſo that a general and conciſe method, equivalent to theirs in accuracy and evidence, that comprehends innumerable theorems in a few general views, may well be eſteemed a valuable invention.

Cavalerius was ſenſible of the difficulties, as well as the advantages that attended his method. He ſpeaks as if he foreſaw that it ſhould be afterwards delivered in an unexceptionable form, that might ſatisfy the moſt ſcrupulous Geometrician; and leaves this *Gordian knot*, as he expreſſes himſelf, to ſome *Alexander*. Its form indeed was ſoon altered, and many improvements were made by the Mathematicians who proſecuted it ſince his time that deſerve to be mentioned with eſteem. But the method ſtill remained liable to ſome exceptions, and was thought to be leſs perfect than that of the ancients on ſeveral grounds.

Sir Isaac Newton accompliſhed what Cavalerius wiſhed for, by inventing the method of Fluxions, and propoſing it in a way that admits of ſtrict demonſtration, which requires the ſuppoſition of no quantities but ſuch as are finite, and eaſily conceived. The computations in this method are the ſame as in the method of infinitesimals; but it is founded on accurate principles, agreeable to the ancient geometry. In it, the premiſſes

and concluſions are equally accurate, no quantities are rejected as infinitely ſmall, and no part of a curve is ſuppoſed to coincide with a right line. The excellency of this method has not been ſo fully deſcribed, or ſo generally attended to, as it ſeems to deſerve; and it has been ſometimes repreſented as on a level in all theſe reſpects with the method of infiniteſimals. The chief deſign of the following treatiſe is, to ſhew its advantages in a clearer and fuller light, and to promote the deſign of the great inventor, by eſtabliſhing the higher geometry on plain principles, perfectly conſiſtent with each other and with thoſe of the ancient Geometricians.

The method of demonſtration which we make moſt uſe of in this treatiſe, was firſt ſuggeſted to us from a particular attention to Sir ISAAC NEWTON's brief reaſoning in that place of his principles of philoſophy where he firſt publiſhed the elements of this doctrine. After the greateſt part of the following treatiſe was writ, we had the pleaſure to obſerve, that Geometricians of the firſt rank had recourſe to it long ago on ſeveral occaſions, as a method of the ſtricteſt kind. Mr. de FERMAT, in a letter to GASSENDUS, and Mr. HUYGENS, in his *Horologium oſcillatorium*, have employed it for completing the demonſtrations of ſome theorems that were propoſed by GALILEUS, and proved by him in a leſs accurate manner; and Dr. BARROW has demonſtrated by it a theorem concerning the tangents of curve lines. The approbation which it appears to have had from ſo good judges, encouraged us to publiſh the following treatiſe; where it is applied for demonſtrating the method of Fluxions. The chief purſuit of Geometricians for ſome time has been to improve their general methods. In proportion as theſe are valuable, it is important that they be eſtabliſhed above all exception: and ſince they ſave us ſo much time and labour, we may allow the more for illuſtrating theſe methods themſelves.

THE

THE ELEMENTS OF THE Method of FLUXIONS,

Demonſtrated after the Manner of the

Ancient Geometricians.

BOOK I.

Of the Fluxions of Geometrical Magnitudes.

CHAP. I.

Of the Grounds of this Method.

1. THE mathematical ſciences treat of the relations of quantities to each other, and of all their affections that can be ſubjected to rule or meaſure. They treat of the properties of figures that depend on the poſition and form of the lines or planes that bound them, as well as thoſe that depend on their magnitude; of the direction of motion, as well as its velocity; of the compoſition and reſolution of quantities, and of every thing of this nature that is ſuſceptible of a regular determination. We enquire

into the relations of things, rather than their inward eſſences, in theſe ſciences. Becauſe we may have a clear conception of that which is the foundation of a relation, without having a perfect or adequate idea of the thing it is attributed to *, our ideas of relations are often clearer and more diſtinct than of the things to which they belong; and to this we may aſcribe in ſome meaſure the peculiar evidence of the mathematics. It is not neceſſary that the objects of the ſpeculative parts ſhould be actually deſcribed, or exiſt without the mind; but it is eſſential, that their relations ſhould be clearly conceived, and evidently deduced: and it is uſeful, that we ſhould chiefly conſider ſuch as correſpond with thoſe of external objects, and may ſerve to promote our knowledge of nature.

2. In our purſuits after knowledge, we ſometimes conſider things as they appear to be in themſelves, ſometimes we judge of them from their cauſes, and ſometimes by their effects. In ordinary enquiries, but eſpecially in philoſophy, we employ one or more of theſe methods according as we find ground for applying them. The two laſt may be no leſs ſatisfactory than the firſt, when there is a ſufficient foundation for them; and by carrying our enquiries to the ſprings and principles of things, our knowledge of them becomes more perfect, and our views more extenſive. In geometry, there are various ways of diſcovering the affections and relations of magnitudes that correſpond to theſe general methods of enquiry. In the common geometry, we ſuppoſe the magnitudes to be already formed, and compare them or their parts, immediately, or by the intervention of others of the ſame kind, to which they have a relation that is already known. In the doctrine which we propoſe to explain and demonſtrate in this treatiſe, we have recourſe to the geneſis of quantities, and either deduce their relations, by comparing the powers which are conceived to generate them; or, by comparing the quantities that are generated, we diſcover the relations of theſe powers and of any quantities that are ſuppoſed to be repreſented by them. The power by which magnitudes are conceived to be generated in geometry, is motion; and therefore we muſt begin with ſome account of it.

* Eſſay concerning the human underſtanding, book 2. chap. 25. § 8.

3. No

3. No quantities are more clearly conceived by us than the limited parts of ſpace and time. They conſiſt indeed always of parts; but of ſuch as are perfectly uniform and ſimilar. Thoſe of ſpace exiſt together; thoſe of time flow continually: but by motion they become the meaſures of each other reciprocally. The parts of ſpace are permanent; but being deſcribed ſucceſſively by motion, the ſpace may be conceived to flow as the time. The time is ever periſhing; but an image or repreſentation of it is preſerved and preſented to us at once in the ſpace deſcribed by the motion.

4. Time is conceived to flow always in an uniform courſe, that ſerves to meaſure the changes of all things. When the ſpace deſcribed by motion flows as the time, ſo that equal parts of ſpace are deſcribed in any equal parts of the time, the motion is uniform; and the velocity is meaſured by the ſpace that is deſcribed in any given time. As this ſpace may be conceived to be greater or leſs, and to be ſuſceptible of all degrees of aſſignable magnitude; ſo may the velocity of the motion by which we ſuppoſe the ſpace to be always deſcribed in a given time. The velocity of an uniform motion is the ſame at any term of the time during which it continues. But motion is ſuſceptible of the ſame variations with other quantities, and the velocity in other inſtances may increaſe or decreaſe while the time increaſes. In theſe caſes, however, the velocity at any term of the time is accurately meaſured by the ſpace that would be deſcribed in a given time, if the motion was to be continued uniformly from that term.

5. Any ſpace and time being given, a velocity is determined by which that ſpace may be deſcribed in that given time: And, converſely, a velocity being given, the ſpace which would be deſcribed by it in any given time is alſo determined. This being evident, it does not ſeem to be neceſſary, in pure geometry, to enquire further what is the nature of this power, affection or mode, which is called *Velocity*, and is commonly aſcribed to the body that is ſuppoſed to move. It ſeems to be ſufficient for our purpoſe, that while a body is ſuppoſed in motion, it muſt be conceived to have ſome velocity or other at any term of the time during which it moves, and that we can demonſtrate accurately

rately what are the meaſures of this velocity at any term, in the enquiries that belong to this doctrine, as will appear in the courſe of this treatiſe; eſpecially ſince it is the buſineſs of geometry, as we have obſerved already, to enquire into the meaſures, rather than unfold the hidden eſſences of things.

6. But perhaps this explication will not be thought ſufficient, and it will be required that we ſhould propoſe a definition of velocity in form. The excellent Dr. BARROW defines it to be the power by which a certain ſpace may be deſcribed in a certain time. Some perhaps may ſcruple to aſcribe power to a body, figure or point in motion. But it is to be obſerved, that it is of no conſequence, in pure geometry, to what the power may be moſt properly attributed. It is indeed generally allowed, that if a body was to be left to itſelf from any term of the time of its motion, and was to be affected by no external influence after that term, it would proceed for ever with an uniform motion, deſcribing always a certain ſpace in a given time: and this ſeems to be a ſufficient foundation for aſcribing, in common language, the velocity to the body that moves, as a power. It is well known, that what is an effect in one reſpect, may be conſidered as a power or cauſe in another; and we know no cauſe in common philoſophy, but what is itſelf to be conſidered as an effect: but this does not hinder us from judging of effects from ſuch cauſes. However, if any diſlike this expreſſion, they may ſuppoſe any mover or cauſe of the motion they pleaſe, to which they may aſcribe the power, conſidering the velocity as the action of this power, or as the adequate effect and meaſure of its exertion, while it is ſuppoſed to produce the motion at every term of the time. We have obſerved already, that the principles of this method are analogous to the general doctrine of powers, or may be conſidered as a particular application of it. As a power which acts continually and uniformly is meaſured by the effect that is produced by it in a given time, ſo the velocity of an uniform motion is meaſured by the ſpace that is deſcribed in a given time. If the action of the power vary, then its exertion at any term of the time is not meaſured by the effect that is actually produced after that term in a given time, but by the effect that would have been produced

ced if its action had continued uniform from that term: and, in the ſame manner, the velocity of a variable motion at any given term of time is not to be meaſured by the ſpace that is actually deſcribed after that term in a given time, but by the ſpace that would have been deſcribed if the motion had continued uniformly from that term. If the action of a variable power, or the velocity of a variable motion, may not be meaſured in this manner, they muſt not be ſuſceptible of any menſuration at all. It will appear afterwards, in the courſe of this treatiſe, that the other principles of this method correſpond with the plain maxims of the general doctrine of powers that are employed by us on every occaſion, and are to be reckoned amongſt the moſt common and evident notions. There are two fundamental principles of this method. The firſt is, That when the quantities which are generated are always equal to each other, the generating motions muſt be always equal. The ſecond is the converſe of the firſt, That when the generating motions are always equal to each other, the quantities that are generated in the ſame time muſt be always equal. The firſt is the foundation of the direct method of fluxions; the ſecond, of the inverſe method. But it is obvious, that they may be conſidered as caſes of theſe two general principles: When the effects produced by two powers are always equal to each other, then (ſuppoſing that no other power of any kind affects their operations) theſe powers muſt be ſuppoſed to act equally at any term of the time; and, converſely, When the actions of two powers are always equal to each other at any term of the time, then the effects produced by them in the ſame time muſt be always equal.

7. This method is ſo well founded, that its rules and operations may be delivered in a way conſiſtent with any general principles that are not repugnant to the moſt evident notions; though it is impoſſible for us, in treating of it, to keep to expreſſions that may appear equally conſiſtent with every ſcheme of metaphyſics. It has been frequently conſidered in a manner agreeable to the principles of thoſe who ſuppoſe quantities to conſiſt of indiviſible or infinitely ſmall elements. We are to proceed upon more ſtrict and rigid principles: but it will be hardly poſſible

for

for us to avoid always ſuch expreſſions as may have been ſome time or other matter of diſpute amongſt philoſophers. Their controverſies concerning continued and diſcrete quantity have not been thought to weaken the evidence of the common geometry. Nor can their diſputes * concerning motion affect the certainty of this method; ſince we have occaſion in it for no more than the moſt obvious notions of ſpace, time, motion and velocity, that cannot be ſaid to yield in clearneſs and evidence to the principles of the common geometry.

8. When we ſuppoſe that a body has ſome velocity or other at any term of the time during which it moves, we do not therefore ſuppoſe that there can be any motion in a term, limit or moment of time, or in an indiviſible point of ſpace: and as we ſhall always meaſure this velocity by the ſpace that would be deſcribed by it continued uniformly for ſome given finite time, it ſurely will not be ſaid that we pretend to conceive motion or velocity without regard to ſpace and time.

9. But to proceed: When any quantity is propoſed, all others of the ſame kind may be conceived to be generated from it; ſuch as are greater than it, by ſuppoſing it to be increaſed; ſuch as are leſs, by ſuppoſing it to be diminiſhed. In the common arithmetic, integer numbers are conceived to be produced by adding a given quantity or unit to itſelf continually, and fractions are produced by ſuppoſing it to be divided into ſuch parts as by a like addition would generate the given quantity itſelf. But in geometry, that all degrees of magnitude may be produced, and in ſuch a way as may found a general method of deriving their affections from their geneſis, we conceive the quantities to be increaſed and diminiſhed, or to be wholly generated by motion, or by a continual flux analogous to it. The quantity that is thus generated, is ſaid to flow, and called *a Fluent.*

10. Lines are generated by the motion of points; ſurfaces, by the motion of lines; ſolids, by the motion of ſurfaces; angles, by the rotation of their ſides; the flux of time being ſuppoſed to be always uniform. The velocity with which a line flows,

* De natura motûs & recta definitione, de cauſis ac differentiis, complura ſubtiliter argutantur Phyſici; quarum ferè Mathematicis nihil cordi vel curæ: ſufficere poteſt his quæ communis ſenſus agnoſcit. *Barrow, lect. geom.* 1.

is

is the ſame as that of the point which is ſuppoſed to deſcribe or generate it. The velocity with which a ſurface flows, is the ſame as the velocity of a given right line, that, by moving parallel to itſelf, is ſuppoſed to generate a rectangle which is always equal to the ſurface. The velocity with which a ſolid flows, is the ſame as the velocity of a given plain ſurface, that, by moving parallel to itſelf, is ſuppoſed to generate an erect priſm or cylinder that is always equal to the ſolid. The velocity with which an angle flows, is meaſured by the velocity of a point, that is ſuppoſed to deſcribe the arch of a given circle, which always ſubtends the angle, and meaſures it. In general, all quantities of the ſame kind (when we conſider their magnitude only, and abſtract from their poſition, figure, and other affections) may be repreſented by right lines, that are ſuppoſed to be always in the ſame proportion to each other as theſe quantities. They are repreſented by right lines in this manner in the Elements, in the general doctrine of proportion, and by right lines and figures in the *Data* of EUCLID *. In this method likewiſe, quantities of the ſame kind may be repreſented by right lines, and the velocities of the motions by which they are ſuppoſed to be generated, by the velocities of points moving in right lines. All the velocities we have mentioned are meaſured, at any term of the time of the motion, by the ſpaces which would be deſcribed in a given time, by theſe points, lines or ſurfaces, with their motions continued uniformly from that term.

11. The velocity with which a quantity flows, at any term of the time while it is ſuppoſed to be generated, is called its *Fluxion* which is therefore always meaſured by the increment or decrement that would be generated in a given time by this motion, if it was continued uniformly from that term without any acceleration or retardation: or it may be meaſured by the quantity that is generated in a given time by an uniform motion which is equal to the generating motion at that term.

12. Time is repreſented by a right line that flows uniformly, or is deſcribed by an uniform motion; and a moment or termination of time is repreſented by a point or termination of

* See the preface to the *Data* by Marinus, near the end.

that line. A given velocity is repreſented by a given line, the ſame which would be deſcribed by it in a given time. A velocity that is accelerated or retarded, is repreſented by a line that increaſes or decreaſes in the ſame proportion. The time of any motion being repreſented by the baſe of a figure, and any part of the time by the correſponding part of the baſe; if the ordinate at any point of the baſe be equal to the ſpace that would be deſcribed, in a given time, by the velocity at the correſponding term of the time continued uniformly, then any velocity will be repreſented by the correſponding ordinate. The fluxions of quantities are repreſented by the increments or decrements deſcribed in the laſt article which meaſure them; and, inſtead of the proportion of the fluxions themſelves, we may always ſubſtitute the proportion of their meaſures.

13. When a motion is uniform, the ſpaces that are deſcribed by it in any equal times are always equal. When a motion is perpetually accelerated, the ſpaces deſcribed by it in any equal times that ſucceed after one another, perpetually increaſe. When a motion is perpetually retarded, the ſpaces that are deſcribed by it in any equal times that ſucceed after one another, perpetually decreaſe.

14. It is manifeſt, converſely, that if the ſpaces deſcribed in any equal times are always equal, then the motion is uniform. If the ſpaces deſcribed in any equal times that ſucceed after one another perpetually increaſe, the motion is perpetually accelerated: For it is plain, that if the motion was uniform for any time, the ſpaces deſcribed in any equal parts of this time would be equal; and if it was retarded for any time, the ſpaces deſcribed in equal parts of this time that ſucceed after one another would decreaſe: both of which are againſt the ſuppoſition. In like manner it is evident, that a motion is perpetually retarded, when the ſpaces that are deſcribed in any equal times that ſucceed after one another perpetually decreaſe. The following Axioms are as evident as that a greater or leſs ſpace is deſcribed in a given time, according as the velocity of the motion is greater or leſs.

AXIOM I.

15. *The ſpace deſcribed by an accelerated motion is greater than the ſpace which would have been deſcribed in the ſame time, if the motion had not been accelerated, but had continued uniform from the beginning of the time.*

AXIOM II.

The ſpace deſcribed by a motion while it is accelerated, is leſs than the ſpace which is deſcribed in an equal time by the motion that is acquired by that acceleration continued uniformly.

AXIOM III.

The ſpace deſcribed by a retarded motion is leſs than the ſpace which would have been deſcribed in the ſame time, if the motion had not been retarded, but had continued uniform from the beginning of the time.

AXIOM IV.

The ſpace deſcribed by a motion while it is retarded, is greater than the ſpace which is deſcribed in an equal time by the motion that remains after that retardation, continued uniformly.

16. Before we proceed to enquire into the fluxions of quantities, it is neceſſary to premiſe the following general Theorems, which contain the grounds of this method. The two firſt are from the treatiſe of ARCHIMEDES concerning ſpiral lines.

THEOREM I.

The ſpaces deſcribed by an uniform motion are in the ſame proportion to each other as the times in which they are deſcribed.

Suppoſe a point to deſcribe the right line AB with an uniform motion. Let it deſcribe the ſpace CD in the time FG, and the ſpace DE in the time GH; then ſhall CD be to DE as FG is to GH.

Let KD and MG be any equimultiples of the lines CD and FG; and let DL and GN be any equimultiples of DE and GH. Becauſe the motion of the point in the line AB is ſuppoſed to be uniform, and it deſcribes CD in the time FG, it will deſcribe any line equal to CD in a time equal to FG, and it will deſcribe KD, that is a multiple of CD, in the time MG, that is an equimultiple of FG. For the ſame reaſon, it will deſcribe DL in the time GN. But, when a motion is uniform, a greater ſpace is always deſcribed in a greater time: ſo that, if the ſpace KD exceed DL, the time MG will exceed GN; if KD be equal to DL, the time MG will be equal to GN; and if KD be leſs than DL, the time MG muſt be leſs than GN. Therefore (by def. 5. lib. 5. Elem.) CD is to DE as FG is to GH; that is, the ſpaces deſcribed by an uniform motion are in the ſame proportion to each other as the times in which they are deſcribed.

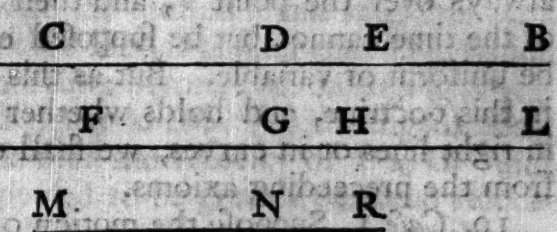
A K C D E L B

M F G H N

THEOREM II.

17. *The ſpaces deſcribed by an uniform motion are to each other in the ſame proportion as the ſpaces deſcribed in the ſame times by any other uniform motion.*

Suppoſe



Suppoſe two points to deſcribe the right lines AB, KL with any uniform motions. Let CD and FG be deſcribed by them in the time MN; and let DE and GH be deſcribed by them in the time NR. Then, by the firſt theorem, the ſpace CD is to the ſpace DE as the time MN is to the time NR; and FG is to GH in the ſame ratio of MN to NR. Therefore CD is to DE as FG is to GH; that is, the ſpaces deſcribed by the firſt motion are in the ſame proportion to each other, as the ſpaces deſcribed in the ſame, or in equal times, by the ſecond motion. The ſpiral of ARCHIMEDES being deſcribed by the compoſition of two uniform motions, one of which is rectilineal, the other circular, he had occaſion, in demonſtrating its properties, to make uſe of no more of the doctrine of motion than theſe two theorems. But, in eſtabliſhing a general method for diſcovering the properties of curvilineal figures, we have occaſion alſo for theſe that follow.

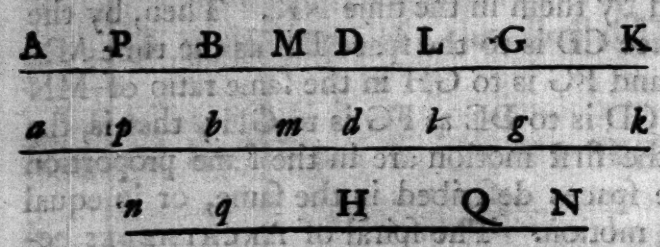
THEOREM III.

18. *If the ſpaces that are deſcribed in the ſame time by two motions, uniform or variable, be always equal to each other, the velocities of theſe motions muſt be equal at any term of the time.*

Suppoſe the points P and *p* to deſcribe the lines AK, *ak* in the ſame time, with motions uniform or varied at pleaſure, but ſo that the ſpace deſcribed by P be always equal to the ſpace deſcribed by *p* in the ſame time. Then ſhall the velocity of *p* at any term or moment of time be equal to the velocity of P at the ſame term or moment. This theorem is ſo evident, that it may ſeem to need no proof. If AK and *ak* be right lines, and we ſuppoſe *ak* to be placed upon AK, the point *p* will be always

always over the point P, and their velocities at the ſame term of the time cannot but be ſuppoſed equal, whether the motions be uniform or variable. But as this is a fundamental theorem in this doctrine, and holds whether the points P and *p* move in right lines or in curves, we ſhall demonſtrate its various caſes from the preceeding axioms.

19. *Caſe* 1. Suppoſe the motion of P to be uniform. Then, ſince equal ſpaces are deſcribed by P in any equal times, it follows, from the ſuppoſition, that equal ſpaces are alſo deſcribed by *p* in any equal times, and that its motion is alſo uniform. But the velocities of uniform motions are equal when equal ſpaces are deſcribed by them in the ſame time; and therefore the velocities of P and *p* are in this caſe always equal to each other.



20. *Caſe* 2. Suppoſe that the motion of P is perpetually accelerated, or that the ſpaces deſcribed by it in any equal parts of time that ſucceed after one another, perpetually increaſe. Then the ſpaces deſcribed by *p* being always equal (by the ſuppoſition) to the ſpaces deſcribed in the ſame times by P, the ſpaces deſcribed by *p* in any equal parts of time that ſucceed each other muſt alſo perpetually increaſe, and the motion of *p* muſt be perpetually accelerated, (by art. 14.) Let P come to D, and *p* to *d*, at the ſame term or moment of time; and their velocities at that term ſhall be equal. For, if they are not equal, ſuppoſe firſt the velocity of *p* to exceed the velocity of P; let DG and *dg* be any equal ſpaces deſcribed after that term by the points P and *p* in the time repreſented by HN. Then, becauſe the motions of theſe points are perpetually accelerated during the time HN, it follows from the firſt axiom, that the ſpace which would be deſcribed in that time by *p* with its motion at *d* continued uniformly, is leſs than *dg*, which is deſcribed by it with an accelerated motion in the ſame time; and it follows from the ſecond axiom, that the ſpace which would be deſcri-

deſcribed by the point P with the motion it has acquired at G continued uniformly for the time HN, is greater than DG, which is deſcribed by it in the ſame time before that motion is acquired. Therefore, ſince *dg* is equal to DG, (by the ſuppoſition,) the velocity of *p* at *d* is leſs than the velocity of P at G. But the velocity of *p* at *d* is ſuppoſed to be greater than the velocity of P at D; and therefore it may be ſuppoſed equal to the velocity of P at ſome intermediate term of the time HN, as when it comes to a point L betwixt D and G. Let *dl* be equal to DL, and let HQ be the time in which P and *p* deſcribe the equal ſpaces DL and *dl* with their accelerated motions. Then, by the firſt axiom, the ſpace which would be deſcribed in the time HQ by the motion of *p* at *d* continued uniformly, is leſs than *dl*, which is deſcribed in the ſame time by its accelerated motion; and, by the ſecond axiom, the ſpace which would be deſcribed in the ſame time HQ by the point P with the motion it has acquired at L continued uniformly, is greater than DL, which was deſcribed by it in the ſame time before that motion was acquired. But, by the ſuppoſition, *dl* is equal to DL; and therefore a leſs ſpace would be deſcribed in the ſame time by the motion of *p* at *d* continued uniformly, than by the motion of P at L continued uniformly: ſo that the velocity of *p* at *d* muſt be leſs than the velocity of P at L. But theſe velocities were ſuppoſed equal; and theſe being contradictory, it appears that the velocity of *p* at *d* is not greater than the velocity of P at D. In the ſame manner it is ſhewn, that the velocity of P at D is not greater than the velocity of *p* at *d*. Therefore the velocities of P and *p* are equal at this or any other term of the time of their motion.

21. The velocities with which the points P and *p* come to D and *d* may be ſhewn to be equal, from the ſame principles, without ſuppoſing their motions to be continued after the term H, by conſidering the ſpaces deſcribed by them before that term. Let BD and *bd* be any equal ſpaces deſcribed by the points P and *p* with their accelerated motions in the time *n*H, before they come to D and *d*; and if their velocities be not then equal, let the velocity of *p* at *d* exceed the velocity of P at D. By the ſecond axiom, the ſpace that would be deſcribed

bed by the motion of P at D continued uniformly in the time *n*H is greater than BD. By the firſt axiom, the ſpace that would be deſcribed in the ſame time *n*H by the motion of *p* at *b* continued uniformly, is leſs than *bd* or BD. Therefore the velocity of P at D is greater than the velocity of *p* at *b*: but it is ſuppoſed to be leſs than the velocity of *p* at *d*; and conſequently, it may be ſuppoſed equal to the velocity of *p* at ſome intermediate term *q* of the time *n*H, when *p* comes to ſome point *m* betwixt *b* and *d*. Let MD be equal to *md*, and the ſpaces MD, *md* will be deſcribed in the ſame time *q*H by the points P and *p*. By the ſecond axiom, the ſpace which would be deſcribed in the time *q*H by the motion of P at D continued uniformly, is greater than MD. By the firſt axiom, the ſpace which would be deſcribed in the ſame time *q*H by the motion of *p* at *m* continued uniformly, is leſs than *md* or MD. Therefore the velocity of P at D is greater than the velocity of *p* at *m*: but they were ſuppoſed equal, and theſe are contradictory. It appears therefore, that the velocity of *p* at *d* is not greater than the velocity of P at D; and in the ſame manner it is ſhewn, that the velocity of P at D is not greater than the velocity of *p* at *d*: and therefore theſe velocities muſt be equal to each other.

22. *Caſe* 3. Suppoſe that the motion of P is perpetually retarded, or that the ſpaces perpetually decreaſe which are deſcribed by it in any equal times that ſucceed each other. Then, by the ſuppoſition, the ſpaces deſcribed by *p* in any equal ſucceeding times alſo decreaſe, and its motion is alſo perpetually retarded. If the velocity of P at D be not equal to the velocity of *p* at *d*, let it firſt be greater. Suppoſe DG and *dg* to be equal ſpaces deſcribed, in any time HN, by the points P and *p* with their motions continued after the term H, at which they are ſuppoſed to come to D and *d*. Becauſe their motions are perpetually retarded, it follows from the third axiom, that the ſpace which would be deſcribed, in the time HN, by *p* with its motion at *d* continued uniformly, is greater than *dg*, which is deſcribed in the ſame time by *p* with a retarded motion; and it follows from the fourth axiom, that the ſpace which would be deſcribed by P with the motion that remains at G continued uni-

uniformly for a time equal to HN, is leſs than DG, (or *dg*,) which is deſcribed in the time HN before the motion of P is reduced to the velocity which it retains at the term G. Therefore the velocity of *p* at *d* is greater than the velocity of P at G: and ſince it is ſuppoſed leſs than the velocity of P at D, it may therefore be ſuppoſed equal to ſome intermediate velocity of P, as to that with which it comes to ſome point L betwixt D and G. Let *dl* be equal to DL; and let theſe ſpaces be deſcribed by the points P and *p* with their retarded motions in the time HQ. Then, by the third axiom, the ſpace that would be deſcribed in the time HQ by the motion of *p* at *d* continued uniformly is greater than *dl*; and, by the fourth axiom, the ſpace that would be deſcribed in the ſame time HQ by the motion of P at L continued uniformly is leſs than DL or *dl*. Therefore the velocity of *p* at *d* is greater than the velocity of P at L: but they were ſuppoſed equal; and theſe are contradictory. It appears, therefore, that the velocity of P at D is not greater than the velocity of *p* at *d*; and ſince it is ſhewn, in the ſame manner, that the velocity of *p* at *d* is not greater than the velocity of P at D, theſe velocities muſt therefore be equal. It is eaſy to ſhew that theſe velocities are equal, by conſidering the ſpaces BD and *bd* that are deſcribed by P and *p* before they come to D and *d*, without ſuppoſing their motions to be continued after that term.

A P B D L G

a p b d l g.

H Q N

23. If the motions of the points P and *p* are ſometimes accelerated and ſometimes retarded, then DG and *dg*, or BD and *bd*, muſt be ſuppoſed to be ſpaces deſcribed by them, either while they are accelerated only, or while they are retarded only. We have chiefly in view, in theſe theorems, ſuch motions as are uniform, and ſuch as are increaſed or diminiſhed by a continued acceleration or retardation. But the demonſtration may be extended to theſe caſes alſo, where the motions are ſuppoſed to be increaſed or diminiſhed at certain terms of the time by aſſignable aug-

ments or decrements at once, by ſuppoſing BD and DG to be the ſpaces deſcribed in the interval of time betwixt two ſuch ſucceeding terms, or in a part of that interval. Or, in the 20th article, inſtead of ſuppoſing, in the latter part of the demonſtration, that the velocity of *p* at *d* is equal to the velocity of P at L, ſome intermediate place betwixt D and G, we may ſuppoſe that it is either equal or greater than the velocity of P at L. In the laſt article, we may ſuppoſe the motion of *p* at *d* to be either equal to the motion of P at L, or leſs than it; and thus the demonſtration may be accommodated to thoſe caſes, when the motions of P and *p* are perpetually increaſing or decreaſing, but not in a continued manner. See below, art. 44, 45 and 46.

THEOREM IV.

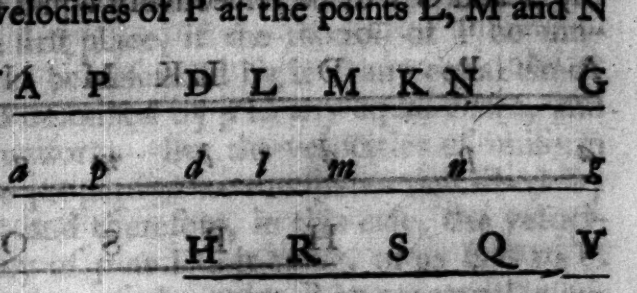
24. *If the velocities of two motions are always equal to each other, the ſpaces deſcribed by them in the ſame time are always equal.*

Suppoſe the points P and *p* to deſcribe the right lines AG and *ag* with any motions, uniform or variable, but ſo as that the velocity of *p* at any term or moment of time be always equal to the velocity of P at the ſame term. Let DG and *dg* be any ſpaces deſcribed by them in the ſame time HV; and DG ſhall be always equal to *dg*.

Caſe 1. If the motions of the points P and *p* are uniform, it is evident, that ſince theſe motions are equal, (by the ſuppoſition,) the ſpaces deſcribed by them in the ſame time muſt be equal; and therefore in this caſe DG is equal to *dg*.

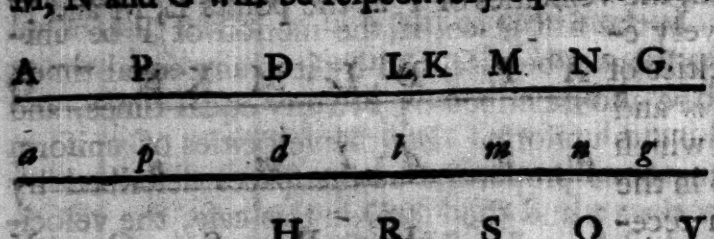
25. *Caſe* 2. Suppoſe the motions of the points P and *p* to be perpetually accelerated while they deſcribe the ſpaces DG and *dg*; and if *dg* be not equal to DG, let it firſt be equal to any line DK leſs than DG. Let the right line HV, which repreſents the time, be divided by a continual biſection into equal parts HR, RS, SQ, QV, till the time repreſented by one of theſe parts, as QV, be leſs than the time in which the point P deſcribes KG. Let DL, LM, MN, NG be the ſpaces deſcribed by P, and *dl*, *lm*, *mn*, *ng* the ſpaces deſcribed by *p*, in the

equal parts of the time repreſented by HR, RS, SQ, QV; and, by the ſuppoſition, the velocities of P at the points L, M and N will be reſpectively equal to the velocities of *p* at the points *l*, *m* and *n*. The ſpace *ng*, which is deſcribed by *p* in the time QV with an accelerated motion, is greater than the ſpace which would be deſcribed in the ſame time by the motion of *p* at *n* continued uniformly, by the firſt axiom. The velocity of *p* at *n* is equal to the velocity of P at N; and, by the ſecond axiom, the ſpace which would be deſcribed in the time QV by the point P with its motion at N continued uniformly, is greater than MN, which was deſcribed by it in an equal time before its velocity at N was acquired. Therefore *ng* is greater than MN. In the ſame manner it appears, that *mn* is greater than LM, and *lm* greater than DL; ſo that *lg* is greater that DN, and *dg* is ſurely greater than DN. But DN is greater than DK; for the time QV is ſuppoſed to be leſs than the time in which KG is deſcribed by the point P; ſo that NG being deſcribed in the time QV, it muſt be leſs than KG, and DN muſt be greater than DK. Therefore *dg* being greater than DN, it muſt be greater than DK: but it was ſuppoſed equal to DK; and theſe are contradictory. It appears, therefore, that the ſpace *dg* is not leſs than DG. In the ſame manner it is ſhewn, that DG is not leſs than *dg*; and therefore theſe ſpaces DG and *dg*, which are deſcribed by the points P and *p* in the ſame time, are equal to each other.



26. *Caſe* 3. Suppoſe the motions of the points P and *p* to be perpetually retarded; and if *dg* be not equal to DG, let it firſt be equal to a line KG leſs than DG. Let the time HV be divided by a continual biſection into the equal parts HR, RS, SQ, QV, till the time HR be leſs than that in which P deſcribes DK. Let DL, LM, MN, NG be the ſpaces deſcribed by P, and *dl*, *lm*, *mn*, *ng* be the ſpaces deſcribed by *p*, in the equal times HR, RS, SQ, QV. Then DL ſhall be leſs than DK, becauſe it is deſcribed by the

point P in a leſs time; and the velocities of P at the points L, M, N and G will be reſpectively equal to the velocities of *p* at the points *l*, *m*, *n* and *g*, by the ſuppoſition. The motions of the points P and *p* being perpetually retarded, it follows, from



the fourth axiom, that *mn* is greater than the ſpace which would be deſcribed in the time SQ (or QV) by the motion of *p* at *n* continued uniformly. But the velocity of *p* at *n* is equal to the velocity of P at N; and, by the third axiom, the ſpace which would be deſcribed in the time QV, by the motion of P at N continued uniformly, is greater than NG, which is deſcribed by P with a retarded motion in the ſame time. Therefore *mn* is greater than NG. In the ſame manner it appears, that *lm* is greater than MN, *dl* greater than LM; and, conſequently, that *dn* is greater than LG: ſo that *dg* is ſurely greater than LG, and therefore greater than KG, which is leſs than LG. But *dg* was ſuppoſed equal to KG; and theſe being contradictory, it follows, that *dg* is not leſs than DG. In the ſame manner it appears, that DG is not leſs than *dg*; and therefore the ſpaces DG and *dg* are equal. By joining theſe caſes together, the theorem is demonſtrated, when the motions of the points P and *p* are ſometimes accelerated, and ſometimes retarded.

THEOREM V.

27. *When the ſpaces deſcribed by two motions in the ſame time are always in an invariable ratio to each other, the velocites of theſe motions are always in the ſame invariable ratio.*

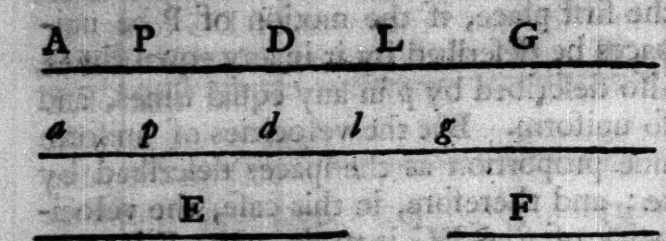
Let DG and *dg* be any two ſpaces deſcribed by the points P and *p* in the ſame time; and let DG be always to *dg* as E is

to

to F. Then the velocity of P at any term of the time ſhall be to the velocity of *p* at the ſame term in the ſame invariable ratio of E to F. In the firſt place, if the motion of P be uniform, ſo that equal ſpaces be deſcribed by it in any equal times, equal ſpaces will be alſo deſcribed by *p* in any equal times, and its motion will be alſo uniform. But the velocities of uniform motions are in the ſame proportion as the ſpaces deſcribed by them in the ſame time; and therefore, in this caſe, the velocity of P is to the velocity of *p* as DG is to *dg*, or as E is to F.

28. *Caſe* 2. If the motion of the point P be perpetually accelerated, then (becauſe the ſpaces deſcribed by P and *p* in the ſame time are always to each other in a given invariable ratio) the ſpaces deſcribed by *p* in equal times perpetually increaſe, and its motion is alſo perpetually accelerated. In the ſame time that the points P and *p* deſcribe the ſpaces DG and *dg* with their accelerated motions, the point P with its motion at G continued uniformly would deſcribe a greater ſpace than DG, (by the ſecond axiom;) and the point *p* with its motion at *d* continued uniformly would deſcribe a leſs ſpace than *dg*, by the firſt axiom. Therefore the velocity of the point P at G is to the velocity of *p* at *d* in a greater ratio than DG is to *dg*, or E is to F. From which it follows, that if the velocity of P at D was to the velocity of *p* at *d* in a leſs ratio than that of E to F, then the velocity of P at ſome intermediate term betwixt D and G might be to the velocity of *p* at *d* in the very ſame ratio as E is to F. But this is impoſſible; for ſuppoſing L to be ſuch a term, let the ſpace DL be to *dl* as E is to F, and theſe ſpaces will be deſcribed by P and *p* in the ſame time, by the ſuppoſition. It follows, from the ſecond axiom, that the point P with its motion at L continued uniformly would deſcribe a greater ſpace than DL, in the ſame time that it deſcribes the ſpace DL with its accelerated motion; and, by the firſt axiom, the point *p* with its motion at *d* continued uniformly would deſcribe a leſs ſpace than *dl* in the ſame time. Therefore the velocity of P at L is to the velocity of *p* at *d* in a greater ratio than DL is to *dl*, or E to F; ſo that the velocity of P at L might be to the velocity of *p* at *d* in the ſame ratio as E is to F, and in a greater ratio at the ſame time; which is abſurd. It appears, there-

therefore, that the ratio of the velocity of P at D to the velocity of *p* at *d* cannot be leſs than that of E to F. Suppoſe now that ratio to be greater than the ratio of E to F. In the ſame time that P and *p* deſcribe DG and *dg* with their accelerated motions, the point P with its motion at D continued uniformly would deſcribe a leſs ſpace than DG, (by the firſt axiom,) and the point *p* with its motion at *g* continued uniformly would deſcribe a greater ſpace than *dg*, by the ſecond axiom: ſo that the velocity of P at D is to the velocity of *p* at *g* in a leſs ratio than DG is to *dg*, or E is to F. Therefore, if the velocity of P at D was to the velocity of *p* at *d* in a greater ratio than that of E to F, the velocity of P at D might be to the velocity of *p* at ſome intermediate term betwixt *d* and *g* in the ſame ratio as E is to F. But this is impoſſible. For ſuppoſing *l* to be ſuch a term, and DL to be to *dl* in the ſame ratio as E is to F, theſe ſpaces would be deſcribed by P and *p* in the ſame time, by the ſuppoſition; and a leſs ſpace than DL would be deſcribed in that time by the motion of P at D continued uniformly, (by the firſt axiom,) but a greater ſpace than *dl* would be deſcribed in the ſame time by the motion of *p* at *l* continued uniformly, by the ſecond axiom; ſo that the velocity of P at D would be to the velocity of *p* at *l* in a leſs ratio than that of DL to *dl*, or of E to F. It appears, therefore, that the velocity of P at D is to the velocity of *p* at *d* neither in a greater nor leſs ratio than that of E to F, but preciſely in this ratio.



29. *Caſe* 3. If the motion of P is perpetually retarded, the motion of *p* muſt alſo be perpetually retarded, becauſe the ſpaces deſcribed by theſe points in equal times decreaſe in the ſame proportion. In this caſe, a leſs ſpace than DG would be deſcribed by the motion of P at G continued uniformly, (by the fourth axiom,) and a greater ſpace than *dg* would be deſcribed by the motion of *p* at *d* continued uniformly, (by the third

third axiom,) in the time that P and *p* with their retarded motions deſcribe DG and *dg*; ſo that the velocity of P at G is to the velocity of *p* at *d* in a leſs ratio than that of DG to *dg*, or of E to F. Therefore, if we ſuppoſe that the velocity of P at D is to the velocity of *p* at *d* in a greater ratio than that of E to F, it follows, that the velocity of P at ſome point betwixt D and G, as L, might be to the velocity of *p* at *d* in the ſame ratio as E is to F. But this is impoſſible. For, ſuppoſing that DL is to *dl* as E is to F, theſe ſpaces will be deſcribed by P and *p* in the ſame time; and a leſs ſpace than DL would be deſcribed in that time by the motion of P at L continued uniformly, (by the fourth axiom;) but a greater ſpace than *dl* would be deſcribed in the ſame time by the motion of *p* at *d* continued uniformly, by the third axiom; ſo that the velocity of P at L is to the velocity of *p* at *d* in a leſs ratio than that of DL to *dl*, or of E to F. It is impoſſible, therefore, that the velocity of P at D can be to the velocity of *p* at *d* in a greater ratio than that of E to F. Nor can that ratio be leſs than the ratio of E to F. For in the ſame time that P and *p* with their retarded motions deſcribe DG and *dg*, a greater ſpace than DG would be deſcribed by the motion of P at D continued uniformly, (by the third axiom,) and a leſs ſpace than *dg* would be deſcribed by the motion of *p* at *g* continued uniformly, by the fourth axiom: ſo that the velocity of P at D is to the velocity of *p* at *g* in a greater ratio than DG is to *dg*, or E is to F. Therefore, if the velocity of P at D was to the velocity of *p* at *d* in a leſs ratio than that of E to F, it might be to the velocity of *p* at ſome point as *l* betwixt *d* and *g* in the ſame ratio as E is to F. But this is impoſſible. For, ſuppoſing that DL is to *dl* as E is to F, the ſpaces DL and *dl* will be deſcribed by P and *p* with their retarded motions in the ſame time; and a greater ſpace than DL would be deſcribed in that time by the motion of P at D continued uniformly, (by the third axiom,) but a leſs ſpace than *dl* would be deſcribed in the ſame time by the motion of *p* at *l* continued uniformly, by the fourth axiom: ſo that the velocity of P at D is to the velocity of *p* at *l* in a greater ratio than that of DL to *dl*, or of E to F. It appears, therefore, that the velocity of P at D is to the velocity

city of p at d in a ratio that is neither greater nor leſs than that of E to F, but is preciſely the ſame ratio.

30. This theorem may be alſo demonſtrated, by conſidering the ſpaces that are deſcribed by the points P and p before they come to D and d, whether their motions be ſuppoſed to be continued after that term or not. By joining theſe caſes together, the demonſtration becomes general; and the ſame obſervation we made in the 23d article is to be applied here.

LEMMA I.

31. *If* A *be to* B *in a greater ratio than* E *is to* F, *and* C *be to* D *in a greater ratio than* E *is to* F; *then ſhall the ſum of the antecedents* A *and* C *be to the ſum of the conſequents* B *and* D *in a greater ratio than that of* E *to* F.

For let G be to B as E is to F, and H to D as E to F; then the ſum of G and H ſhall be to the ſum of B and D as E is to F. But A is greater than G, becauſe A is to B in a greater ratio than G is to B; and C is greater than H, becauſe C is to D in a greater ratio than H is to D. Therefore the ſum of A and C is greater than the ſum of G and H; and, conſequently, the ſum of A and C is to the ſum of B and D in a greater ratio than E is to F. In the ſame manner it appears in general, that if there be any number of ratios, each greater than the ratio of E to F, the ſum of all the antecedents ſhall be to the ſum of all the conſequents in a greater ratio than that of E to F.

32. It appears, in the ſame manner, that if A be to B in a leſs ratio than E is to F, and C be alſo to D in a leſs ratio than E is to F; then the ſum of A and C ſhall be to the ſum of B and D in a leſs ratio than that of E to F: and, in general, if there be any number of ratios, each leſs than that of E to F, then the ſum of all the antecedents ſhall be to the ſum of all the conſequents in a leſs ratio than that of E to F.

THE-

THEOREM VI.

33. *When the velocities of any motions are always to each other in an invariable ratio, the ſpaces deſcribed by them in the ſame time are always in the ſame ratio.*

Suppoſe that the velocity of the point P is always to the velocity of *p* in the invariable ratio of E to F; let DG and *dg* be any ſpaces deſcribed by theſe points in the ſame time HV. Then ſhall DG be to *dg* as E is to F. This is the converſe of the preceeding theorem, and may be demonſtrated by it and the fourth theorem: but it may be alſo demonſtrated immediately from the axioms in the following manner. In the firſt place, if the motions of P and *p* are uniform, it is evident, that the ſpaces deſcribed by them in the ſame time are in the ſame proportion as the velocities of the motions; and therefore, in this caſe, DG is to *dg* as E is to F.

34. *Caſe* 2. Suppoſe the motions of P and *p* to be perpetually accelerated; and, if the ratio of DG to *dg* be greater than that of E to F, let DK be to *dg* as E is to F, and DK ſhall be leſs than DG. Let the time HV be divided by a continual biſection into the equal parts HR, RS, SQ, QV, till the time QV be leſs than that in which P deſcribes KG. Let DL, LM, MN, NG be the ſpaces deſcribed by P, and *dl*, *lm*, *mn*, *ng* be the ſpaces deſcribed by *p*, in theſe equal times HR, RS, SQ, QV. Then ſhall NG be leſs than KG, becauſe it is deſcribed by P in a leſs time than KG; and the velocities of P at L, M, N and G, will be to the velocities of *p* at *l*, *m*, *n* and *g* reſpectively, as E is to F, by the ſuppoſition. It follows, from the ſecond axiom, that MN is leſs than the ſpace which would be deſcribed, in the time QV, by the motion of P at N continued uni-

A P D L M K N G

a p d l m k n g

H R S Q V

E F

uniformly; and, by the firſt axiom, *ng* is greater than the ſpace which would be deſcribed in the ſame time by the motion of *p* at *n* continued uniformly. Therefore MN is to *ng* in a leſs ratio than the velocity of P at N is to the velocity of *p* at *n*; that is, in a leſs ratio than E is to F. In the ſame manner, LM is to *mn*, DL to *lm*, and, conſequently, (by art. 32.) DN to *lg* in a leſs ratio than that of E to F. Therefore DN is ſurely to *dg* in a leſs ratio than E is to F, or DK to *dg*; and, conſequently, DN is leſs than DK. But DN is greater than DK, becauſe NG is leſs than KG; and theſe are contradictory. Therefore DG is to *dg* in a ratio that is not greater than that of E to F. If it be ſaid, that DG is to *dg* in a leſs ratio than that of E to F; let DG be to *dk* as E is to F, and *dk* will be leſs than *dg*. Suppoſe the time HV to be divided into the equal parts HR, RS, SQ, QV, till QV be leſs than the time in which *p* deſcribes *kg*. Let DL, LM, MN, NG be the ſpaces deſcribed by P, and *dl*, *lm*, *mn*, *ng* the ſpaces deſcribed by *p*, in the equal times HR, RS, SQ, QV. Then, ſince NG is greater than the ſpace which would be deſcribed, in the time QV, by the motion of P at N continued uniformly, (by the firſt axiom;) and *mn* is leſs than the ſpace which would be deſcribed, in the ſame time, by the motion of *p* at *n* continued uniformly, (by the ſecond axiom:) it follows, that NG is to *mn* in a greater ratio, than that of the velocity of P at N to the velocity of *p* at *n*, which (by the ſuppoſition) is the ratio of E to F. In the ſame manner, MN is to *lm*, LM to *dl*, and therefore LG to *dn* (by art. 31.) in a greater ratio than that of E to F. Therefore DG is ſurely in a greater ratio to *dn*, than that of E to F, or that of DG to *dk*; and, conſequently, *dn* is leſs than *dk*. But the time QV, in which *p* deſcribes *ng*, was ſuppoſed to be leſs than the time in which *p* deſcribes *kg*: therefore *ng* is leſs than *kg*, and *dn* is greater than *dk*: and theſe being contradictory, it fol-

A P D L M K N G

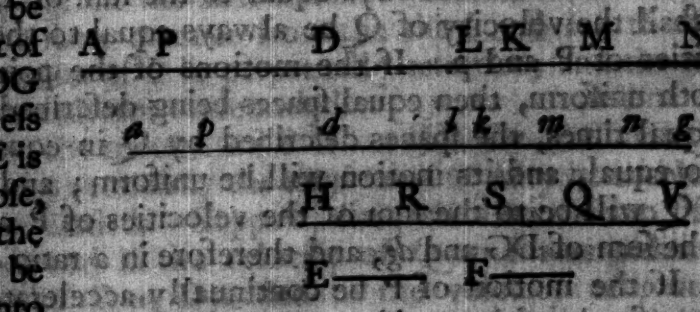
a p d l m k n g

H R S Q V

E—— F——

follows, that the ratio of DG to *dg* is not leſs than that of E to F. It appears, therefore, that DG is to *dg* as E is to F.

35. *Caſe* 3. Suppoſe the motions of P and *p* to be perpetually retarded; and, if DG be to *dg* in a greater ratio than that of E to F, let KG be to *dg* as E is to F, and KG will be leſs than DG. Let the time HV be divided into equal parts, HR, RS, SQ, QV, till HR become leſs than the time in which P deſcribes DK; let DL, LM, MN, NG be deſcribed by the point P, and *dl*, *lm*, *mn*, *ng* be deſcribed by the point *p*, in theſe equal times. Then, becauſe DL is deſcribed by P in a leſs time than DK, DL is leſs than DK, and LG is greater than KG. But LG muſt be ſuppoſed to be leſs than KG: For it eaſily appears, from the third and fourth axioms, that NG is to *ng* in a leſs ratio than that of the velocity of P at N to the velocity of *p* at *n*, or that of E to F; and, in the ſame manner, it appears, that MN is to *lm*, LM to *dl*, and conſequently (by art. 32.) LG to *dn*, in a leſs ratio than E is to F: ſo that LG is ſurely to *dg* in a leſs ratio than E to F, or KG to *dg*: From which it follows, that LG is leſs than KG. If the ratio of DG to *dg* be ſaid to be leſs than that of E to F, let DG be to *kg* (leſs than *dg*) as E is to F. Suppoſe, as before, the time HV to be ſubdivided into the equal parts HR, RS, SQ, QV, till HR become leſs than the time in which *p* deſcribes *dk*; and the ſpaces DG and *dg* being ſubdivided as formerly, *dl* ſhall be leſs than *dk*, and *lg* greater than *kg*. But *lg* muſt be leſs than *kg*. For it appears, from the fourth axiom, that DL is greater than the ſpace which would be deſcribed, in the time HR, by the motion of P at L continued uniformly; and *lm* is leſs than the ſpace which would be deſcribed, in the ſame time, by the motion of *p* at *l* continued uniformly, by the third axiom: ſo that DL is to *lm* in a greater ratio, than that of the velocity of P at L to the veloci-



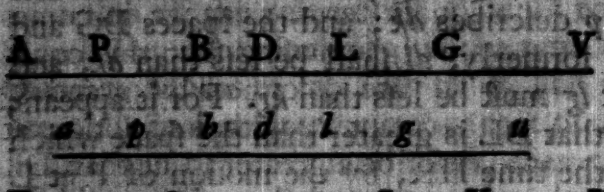
ty of p at l, or that of E to F. In the same manner, LM is to lm, MN is to mg, and consequently (by art. 31.) DN to lg, in a greater ratio than E is to F. Therefore DG is to lg in a greater ratio than E is to F, or DG to kg; and, consequently, lg is less than kg. Thus it appears, that when the velocities of the points P and p are always to each other in any invariable ratio, the spaces described by them in the same time are always in the same ratio.

THEOREM VII.

36. *When the space described by a motion is always equal to the sum of the spaces described in the same time by any other motions, the velocity of the first motion is always equal to the sum of the velocities of the other motions.*

Let the three points P, p and Q move in the lines AV, au and EF, and describe the spaces DG, dg and IH in the same time; and let IH be always equal to the sum of DG and dg: then shall the velocity of Q be always equal to the sum of the velocities of P and p. If the motions of the points P and p are both uniform, then equal spaces being described by them in any equal times, the spaces described by Q in equal times will be also equal, and its motion will be uniform; and the velocity of Q will be to the sum of the velocities of P and p as IK is to the sum of DG and dg, and therefore in a ratio of equality.

37. If the motion of P be continually accelerated, and the motion of p be either uniform or accelerated, the motion of Q is also continually accelerated, (by art. 14.) and if the sum of the velocity of P at D, added to the velocity of p at d, was greater than the velocity of Q at I, this sum might be supposed equal to the velocity of



Q

Q at ſome ſubſequent term, as when it comes to K. Let P and *p* deſcribe DL and *dl* in the ſame time that Q deſcribes IK; and it follows, from the laſt article, that the point Q, with its motion at K continued uniformly, would in this time deſcribe a ſpace equal to the ſum of the ſpaces that would be deſcribed, in the ſame time, by the points P and *p*, with their motions at D and *d* continued uniformly, that is, a ſpace leſs than the ſum of DL and *dl*, (by the firſt axiom;) and therefore leſs than IK. But the point Q, with its motion at K continued uniformly, would deſcribe a greater ſpace than IK in that time, by the ſecond axiom; and theſe being contradictory, it follows, that the velocity of Q at I is not leſs than the ſum of the velocities of P at D and *p* at *d*. If the velocity of Q at I was greater than the ſum of theſe velocities, then its velocity at ſome preceeding term, as when it arrived at *k*, might be equal to that ſum; and, ſuppoſing BD, *bd* and *k*I to be deſcribed by P, *p* and Q in the ſame time, the point Q, with its motion at *k* continued uniformly, might deſcribe a ſpace greater than the ſum of BD and *bd* (or *k*I) in the time it deſcribes *k*I with its accelerated motion, againſt the firſt axiom. This theorem is demonſtrated in the ſame manner from the third and fourth axioms, when the motion of P is perpetually retarded, and the motion of *p* is either uniform or retarded.

38. When the motion of P is continually accelerated, but the motion of *p* retarded, the motion of Q may be accelerated, uniform, or retarded. Suppoſe, firſt, that the motion of Q is accelerated; and if the velocity of Q at I was greater than the velocity of P at D added to the velocity of *p* at *d*, it might be ſuppoſed equal to the velocity of *p* at *d* added to the velocity of P at ſome ſubſequent term, as when it comes to L. But this is impoſſible. For, ſuppoſing that this could be admitted, and that *dl* and IK are deſcribed by *p* and Q in the time P deſcribes DL, the point P, with its motion at L continued uniformly, would deſcribe a greater ſpace than DL in that time, (by the ſecond axiom;) and the point *p*, with its motion at *d* continued uniformly, would deſcribe a greater ſpace than *dl* in the ſame time, (by the third axiom:) ſo that the point Q, with its motion at I continued uniformly, would deſcribe

ſcribe a greater ſpace than the ſum of DL and *dl* (or IK) in that time. But, becauſe the motion of Q is ſuppoſed to be continually accelerated, it follows, from the firſt axiom, that the point Q, with its motion at I continued uniformly, would deſcribe a leſs ſpace than IK in that time; and theſe being contradictory, it appears, that the velocity of Q at I is not equal to the velocity of P at L added to the velocity of *p* at *d*, and cannot exceed the ſum of the velocity of P at D added to the velocity of *p* at *d*. If the velocity of Q at I was leſs than this ſum, it might be equal to the velocity of *p* at *d* added to the velocity of P at ſome preceeding term of the time, as when it came to B. But this is impoſſible. For, if it could be admitted, then, ſuppoſing *bd* and *kI* to be deſcribed by *p* and Q while P deſcribes BD, the point P, with its motion at B continued uniformly, would deſcribe a leſs ſpace than BD (by the firſt axiom) in that time; and the point *p*, with its motion at *d* continued uniformly, would deſcribe a leſs ſpace than *bd* (by the fourth axiom) in the ſame time: ſo that the point Q, with its motion at I continued uniformly, would deſcribe in this time a ſpace leſs than the ſum of BD and *bd*, or *kI*. But the motion of Q being continually accelerated, the point Q, with its motion at I continued uniformly, would deſcribe a greater ſpace than *kI* in that time, by the ſecond axiom: and theſe being contradictory, it is evident, that the velocity of Q at I is not equal to the velocity of P at B added to the velocity of *p* at *d*, and cannot be leſs than the velocity of P at D added to the velocity of *p* at *d*. If the motion of Q be uniform, while the motion of P is accelerated and the motion of *p* is retarded, it is demonſtrated, in the ſame manner, that the velocity of Q at I is equal to the velocity of P at D added to the velocity of *p* at *d*.

39. If the velocity of Q be continually retarded, (the reſt remaining as in the laſt article,) and its velocity at I was greater than the velocity of P at D added to the velocity of *p* at *d*, it might be equal to the velocity of *p* at ſome preceeding term of the time, as when it came to *b*, added to the velocity of P at D. But this is impoſſible. For, ſuppoſing that this is admitted, and that P and Q deſcribe BD and *kI* in the time *p* deſcribes *bd*, the point *p*, with its motion at *b* continued uniformly,

ly, would deſcribe in that time a ſpace greater than *bd*, (by the third axiom;) the point P, with its motion at D continued uniformly, would deſcribe a greater ſpace than BD in the ſame time, (by the ſecond axiom:) and therefore the point Q, with its motion at I continued uniformly, would deſcribe in this time a ſpace greater than the ſum of BD and *bd*, or *kI*. But the motion of Q being continually retarded, it would deſcribe a leſs ſpace than *kI* with its motion at I continued uniformly in that time, by the fourth axiom: and theſe are contradictory. If the velocity of Q at I was leſs than the velocity of P at D added to the velocity of *p* at *d*, it might be ſuppoſed equal to the velocity of P at D added to the velocity of *p* at ſome ſubſequent term of the time, as when it comes to *l*; and (by the third and fourth axioms) it might deſcribe, with its motion at I continued uniformly, a leſs ſpace than the ſum of DL and *dl*, or IK, in the ſame time that by its retarded motion it deſcribes IK; but it would deſcribe, with the ſame uniform motion, in the ſame time, a greater ſpace than IK, by the third axiom: and theſe are contradictory. Therefore the velocity of Q at I is equal to the velocity of P at D added to the velocity of *p* at *d*.

A P B D L G V

a p b d l g v

E Q *k* I K H F

40. If the ſpace deſcribed by Q be always equal to the ſum of the ſpaces deſcribed in the ſame time by three or more points, it is eaſy, from what has been ſhewn, to extend the demonſtration to theſe caſes, by ſubſtituting always one point in place of two: And it appears, in general, that the velocity of Q at any term of the time, is equal to the ſum of the velocities of all the other points at the ſame term.

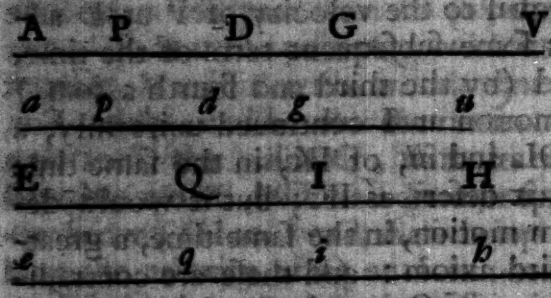
41. It follows, from what has been demonſtrated, that when the ſpace deſcribed by any point *p* is always equal to the difference of the ſpaces deſcribed in the ſame time by the points Q and P, then the velocity of *p* is always equal to the difference of the velocities of theſe points.

THE-

THEOREM VIII.

42. *When the velocity of a motion is always equal to the ſum of the velocities of any other motions, the ſpace deſcribed by it is always equal to the ſum of the ſpaces deſcribed in the ſame time by theſe other motions.*

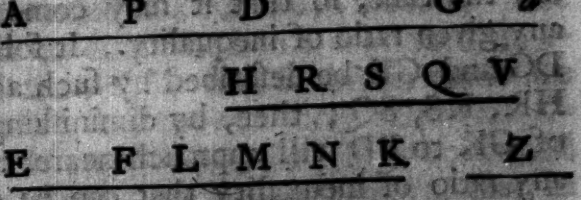
Let the velocity of the point Q be always equal to the ſum of the velocities of the points P and *p*; and let IH, DG and *dg* be any ſpaces deſcribed by Q, P and *p* in the ſame time: then ſhall IH be equal to the ſum of DG and *dg*. This theorem may be demonſtrated immediately from the axioms, in the ſame manner as the fourth and ſixth; but more briefly thus: Suppoſe the point *q* to deſcribe, upon the line *ef*, a ſpace *ih* always equal to the ſum of DG and *dg*, then (by the ſeventh theorem) the velocity of *q* muſt be always equal to the ſum of the velocities of P and *p*, and therefore is equal to the velocity of Q. From which it follows, by the fourth theorem, that IH is always equal to *ih*; and therefore IH is equal to the ſum of DG and *dg*.



43. It follows, from this theorem, that when the velocity of any point *p* is always equal to the difference of the velocities of two other points Q and P, the ſpace *dg* deſcribed by *p* is always equal to the difference betwixt IH and DG, which are deſcribed in the ſame time by the points Q and P.

44. In demonſtrating theſe theorems, we have ſuppoſed that every motion is either uniform, continually accelerated, or continually retarded; or that the time may be diſtinguiſhed into parts, during each of which the motion is reducible to one or other

other of thoſe kinds. For though the velocity of a motion may, at certain terms of the time, be increaſed or diminiſhed at once by a given or aſſignable quantity; it is impoſſible that a motion can be increaſed or diminiſhed in this manner perpetually, or at every term of the time. If ſuch a motion could be ſuppoſed, its velocity would exceed all conceivable velocities in the leaſt time that could be aſſigned. This, if it ſeem to need a proof, may be demonſtrated in the following manner. Let the point P deſcribe any ſpace DG, upon the line A*a*, in the time HV; and let its velocity at G be to its velocity at D in any aſſignable ratio, as in that of EK to EF. The velocity of P may be ſuppoſed, at certain terms of the time, to be increaſed at once, by a quantity that may be in the ſame ratio to the velocity of P at D, as a given line Z is to EF; but it cannot be ſuppoſed to be increaſed in every aſſignable part of the time by this quantity, how minute ſoever it may be. For, ſuppoſing that this could be admitted, let FK be divided by a continual biſection into equal parts, FL, LM, MN, NK, till each of theſe parts become leſs than Z; and let HV, which repreſents the time, be divided into the ſame number of equal parts, HR, RS, SQ, QV. Then, by what we have ſuppoſed, the velocity of P muſt be increaſed, in the time HR, by a quantity that is to the velocity of P at D as Z is to EF; and therefore, ſince Z is greater than FL, the velocity of P at the end of the time HR, is to its velocity at the beginning of that time in a greater ratio than EL is to EF. In like manner, the velocity of P being increaſed in each of the ſubſequent times RS, SQ, QV, by the quantity repreſented by Z, which is greater than LM, MN or NK; it follows, that the velocity of P at the end of the time HV, (that is, when it is ſuppoſed to come to G,) is to the velocity it had at the beginning of that time, when it was at D, in a greater ratio than EK is to EF: but the ratio of theſe velocities was ſuppoſed to be the ſame as that of EK to EF; and theſe are contradictory. It appears, in the



 ſame

ſame manner, that the velocity of P cannot be ſuppoſed to decreaſe, ſo as to be diminiſhed by a given quantity how minute ſoever, repreſented by Z, at every term of the time, or in every aſſignable part of it.

45. When a motion is accelerated or retarded in a continued manner, it is evident, that, from any given term, a part of the time may be taken ſo ſmall, that the difference of the velocities at the beginning and end of this time, may be leſs than the difference of any given unequal velocities; and that the ratio of thoſe velocities approaches to a ratio of equality, by diminiſhing the time, ſo that it may come nearer to that ratio than any given ratio of inequality. It follows from this, that if BD, DG and GK be deſcribed by ſuch a motion in the equal times HR, RS, SQ; then, by diminiſhing the time HQ, the ratio of GK to BD will approach nearer to a ratio of equality than any ratio of inequality that can be aſſigned. For, if the motion be accelerated, BD muſt be greater than the ſpace which would be deſcribed in the time HR by the motion at B continued uniformly, (by the firſt axiom;) and GK is leſs than the ſpace which would be deſcribed, in the ſame time, by the motion at K continued uniformly, (by the ſecond axiom:) ſo that GK is greater than BD, but in a leſs ratio than the velocity at K is greater than the velocity at B; and therefore in a ratio, that, by diminiſhing the time HQ, may approach nearer to a ratio of equality, than any aſſignable ratio of inequality. If the motion of P be retarded in a continued manner, it appears, from the third and fourth axioms, that the ratio of BD to GK is leſs than the ratio of the velocity at B to the velocity at K; ſo that, by diminiſhing the time HQ, the ratio of BD to GK may approach nearer to a ratio of equality, than any given ratio of inequality.

A P B D G K *a*

H R S Q

46. It follows, converſely, that if the ratio of GK to BD approach to a ratio of equality by diminiſhing the time HQ, ſo as to come nearer to it than any given ratio of inequality, and this obtain whatever part of the time be repreſented by HQ; then the motion muſt be increaſed or diminiſhed in a con-

continued manner. For, if the motion perpetually increaſe, then the velocity at G is to the velocity at D in a leſs ratio, than GK is to BD, (by the firſt and ſecond axioms;) and therefore in a ratio that (by the ſuppoſition) may come nearer to a ratio of equality, than any given ratio of inequality: ſo that the motion muſt be accelerated in a continued manner. If the motion perpetually decreaſe, then the velocity at D is to the velocity at G in a leſs ratio than BD is to GK, (by the third and fourth axioms;) and therefore in a ratio that may approach to a ratio of equality nearer than any given ratio of inequality: ſo that the motion is retarded in a continued manner. The motions that are increaſed or diminiſhed in a continued manner, are thus diſtinguiſhed from thoſe that at certain terms are increaſed by a given quantity, but betwixt thoſe terms are either uniform, or are accelerated or retarded in a continued manner. It appears eaſily, from what has been ſhewn, that, in the third and fifth theorems, if the motion of P be accelerated or retarded in a continued manner, the motion of *p* is alſo accelerated or retarded in a continued manner; and that, in the ſeventh theorem, when the motions of P and *p* are accelerated or retarded in a continued manner, the motion of Q is accelerated or retarded in the ſame manner.

THEOREM IX.

47. *When a point* P *deſcribes a line* Aa *with a motion of any kind, and another point p deſcribes the ſame ſpaces on this line* Aa, *in equal times, but in a contrary order, and with an oppoſite direction; the velocities of theſe points at any given term of the line are equal.*

Suppoſe that the point P proceeding from A towards *a* deſcribes any ſpace DL in any time HQ; and that the point *p*, in moving from *a* towards A, deſcribes always that ſpace DL in the ſame time HQ in which it was deſcribed by P: Then the velocity of *p* at D muſt be equal to the velocity of P at D.

When the motion of the point P is uniform, it follows, from the ſuppoſition, that the motion of *p* is alſo uniform, and that their velocities are always equal. Let the motion of P be continually accelerated; and it follows, from the ſuppoſition, that the motion of *p* is continually retarded. If the velocity of the point *p* at D was greater than the velocity of P at D, it might be ſuppoſed equal to the velocity of P at ſome ſubſequent term of the time, as when it comes to L. But this is impoſſible: For the point P, with its motion at L continued uniformly, would deſcribe a greater ſpace than DL, in the time HQ, (by the ſecond axiom;) and the point *p*, with its motion at D continued uniformly, would deſcribe a leſs ſpace than DL, in that time, (by the fourth axiom:) ſo that the velocity of *p* at D is leſs than the velocity of P at L. If the velocity of P at D was greater than the velocity of *p* at D, it might be ſuppoſed equal to the velocity of *p* at ſome term before *p* came to D, as to its velocity at L. But this alſo is impoſſible: For the point P, with its motion at D continued uniformly, would deſcribe a leſs ſpace than DL in the time HQ, (by the firſt axiom;) and the point *p*, with its motion at L continued uniformly, would deſcribe a greater ſpace than LD in the ſame time, (by the third axiom:) ſo that the velocity of *p* at L is greater than the velocity of P at D. It appears, therefore, that the velocity of *p* at any term of the line A*a*, as D, is neither greater nor leſs than the velocity of P at the ſame term, but equal to it. If the motion of P is continually retarded, the motion of *p* muſt be continually accelerated; and the demonſtration is the ſame as in the former caſe.

A P D L *p* *a*

H Q

48. In the following articles, we ſuppoſe, that while the points P and *p* deſcribe the line A*a*, the points M and *m* deſcribe the line E*e*; and that EM is determined in any regular manner from AP, ſo that when AP is equal to the abſciſs of any figure, EM is always equal to the correſponding ordinate. Then, if E*m* be determined in the ſame manner from A*p*, it is evident, that when A*p* becomes equal to AP, E*m* becomes equal

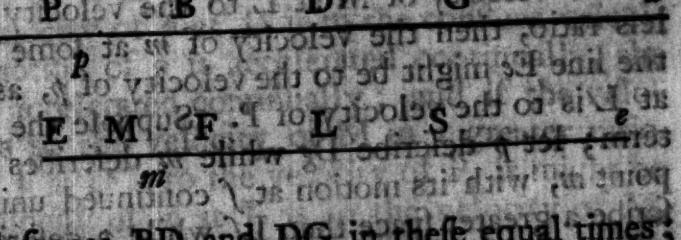
to

to EM; and that, according as AP is greater or leſs than A*p*, EM is greater or leſs than E*m*. If P and *p* come together to any point D, then M and *m* ſhall come at the ſame time to ſome point L upon the line E*e*; and, according as DP is greater or leſs than D*p*, LM is greater or leſs than L*m*. It is alſo evident, that the motion of P being uniform, if the motion of M be accelerated or retarded in a continued manner; then, the motion of *p* being alſo uniform, the motion of *m* muſt be accelerated or retarded in a continued manner. If this ſeem to need any proof, it may be eaſily deduced from the 45th and 46th articles.

THEOREM X.

49. *The motions of the points* P *and* *p* *in the line* A*a* *being uniform, let* EM *be always determined in any regular manner from* AP, *and* E*m* *be determined in the ſame manner from* A*p*; *then the velocity of the point* M, *at any term of the line* E*e*, *ſhall be to the velocity of the point* *m*, *at the ſame term of that line, as the velocity of* P *is to the velocity of* *p*.

Let the points P and *p* come together to D; let M and *m* come at the ſame time to L: and the velocity of M at L ſhall be to the velocity of *m* at L, as the velocity of the uniform motion of P is to the velocity of the uniform motion of *p*. If the motion of M is uniform, the motion of *m* is alſo uniform. For, if the point *m* deſcribe the ſpaces FL, LS in any equal times, the point *p* will deſcribe equal ſpaces BD and DG in theſe equal times; and, the motion of P being uniform, it will deſcribe the ſame ſpaces in equal times. But while P deſcribes BD and DG, M deſcribes FL and LS with its uniform motion; and therefore FL



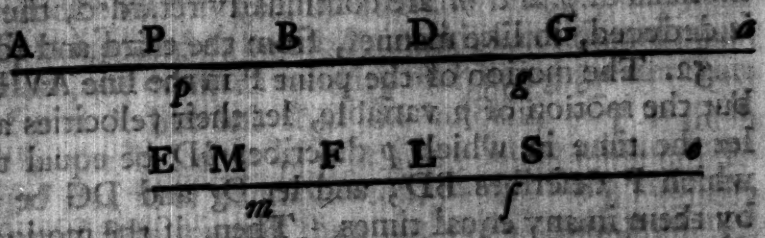
is

is equal to LS: and the ſpaces being equal which are deſcribed by *m* in any equal times, its motion muſt be uniform. The velocity of M is to the velocity of P as FL is to BD; and the velocity of *m* is to the velocity of *p* in the ſame ratio. Therefore the velocity of M is to the velocity of *m* as the velocity of P is to the velocity of *p*.

50. If the motion of M is continually accelerated, the motion of *m* is alſo accelerated continually. In this caſe, if the velocity of *m* at L was to the velocity of *p* in a greater ratio, than the velocity of M at L is to the velocity of P; then the velocity of *m* at L might be to the velocity of *p*, as the velocity of M at ſome ſubſequent term of the line *Ee* is to the velocity of P. Suppoſe the point S to be ſuch a term, and let P deſcribe DG while M deſcribes LS. Then, becauſe the motion of the point M is continually accelerated, it would deſcribe a greater ſpace than LS with its motion at S continued uniformly, in the time P deſcribes DG, (by the ſecond axiom.) From which it follows, that the velocity of M at S is to the velocity of P in a greater ratio than LS is to DG; and therefore the velocity of *m* at L is to the velocity of *p* in a ratio that is alſo greater than that of LS to DG. But, in the ſame time that *p* deſcribes DG with an uniform motion, the point *m* deſcribes LS with an accelerated motion; and it would deſcribe a leſs ſpace than LS with its motion at L continued uniformly, in this time, (by the firſt axiom.) Therefore the velocity of *m* at L is to the velocity of *p* in a leſs ratio than LS is to DG: and theſe being contradictory, it appears, that the ratio of the velocity of *m* at L to the velocity of *p*, is not greater than the ratio of the velocity of M at L to the velocity of P. If it was a leſs ratio, then the velocity of *m* at ſome ſubſequent term of the line *Ee* might be to the velocity of *p*, as the velocity of M at L is to the velocity of P. Suppoſe the point *ſ* to be that term; let *p* deſcribe *Dg* while *m* deſcribes *Lſ*: and ſince the point *m*, with its motion at *ſ* continued uniformly, would deſcribe a greater ſpace than *Lſ* while *p* deſcribes *Dg* with its uniform motion, (by the ſecond axiom;) it follows, that the velocity of *m* at *ſ* is to the velocity of *p* in a greater ratio than *Lſ* is to *Dg*: and therefore the velocity of M at L is to the

velocity

velocity of P in a ratio that is alſo greater than that of L*ſ* to D*g*. But while P deſcribes D*g* with an uniform motion, the point M deſcribes L*ſ* with an accelerated motion; and it would deſcribe a leſs ſpace than L*ſ* with its motion at L continued uniformly, in this time, (by the firſt axiom.) Therefore the velocity of M at L is to the velocity of P in a leſs ratio than L*ſ* is to D*g*. And theſe being contradictory, it appears, that the velocity of *m* at L is to the velocity of *p* neither in a greater nor in a leſs ratio, than the velocity of M at L is to the velocity of P. Therefore the velocity of M at L is to the velocity of *m* at L, as the velocity of P is to the velocity of *p*.



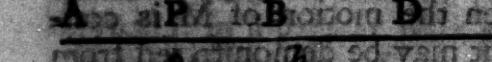
In like manner, this theorem is demonſtrated from the third and fourth axioms, when the motion of M is continually retarded. In either caſe, it may be demonſtrated from the ſame principles, by conſidering the ſpaces deſcribed by the points M and *m* before they come to L, whether their motions be continued after that term or not. An example of the manner how this is done, was given in the 21ſt article.

51 It is demonſtrated, in the ſame manner, that EM being always determined from AP in any regular manner, and E*m* being determined from A*p* in the ſame manner; if the motions of P and M be uniform, and M come to L when P comes to D, then the velocity of *m* at L ſhall be to the conſtant velocity of M, as the velocity of *p* at D is to the conſtant velocity of P. If the motion of *p* is accelerated continually, the motion of *m* is alſo accelerated continually: and if the velocity of *m* at L was to the velocity of M in a greater ratio, than the velocity of *p* at D is to the velocity of P, it might be ſuppoſed that the velocity of *m* at L is to the velocity of M, as the velocity of *p* at *g* is to the velocity of P; that is, (ſuppoſing DG, D*g*, LS, L*ſ* to be deſcribed by the points P, *p*, M, *m* reſpectively, in the ſame time,) in a greater ratio than D*g* is to DG, (by the ſecond axiom,) or (becauſe the motions of P

and

and M are uniform, and Dg is to DG as Lf is to LS) in a greater ratio than Lf is to LS. But the velocity of m at L is to the velocity of M in a leſs ratio than Lf is to LS, (by the firſt axiom;) and theſe being contradictory, it appears, that the ratio of the velocity of m at L to the velocity of M is not greater, than the ratio of the velocity of p at D to the velocity of P. In the ſame manner, it is ſhewn, that it is not a leſs ratio; and therefore the velocity of m at L is to the velocity of M, as the velocity of p at D is to the velocity of P. When the motions of p and m are continually retarded, the demonſtration is deduced, in like manner, from the third and fourth axioms.

52. The motion of the point P in the line AV being uniform, but the motion of p variable, let their velocities at D be equal; let the time in which p deſcribes bD be equal to the time in which P deſcribes BD; and let Dg and DG be alſo deſcribed by them in any equal times. Then, if the motion of p be perpetually accelerated, bD ſhall be always leſs than BD, (by the ſecond axiom,) becauſe bD is deſcribed by p with an accelerated motion, and BD is deſcribed in an equal time by P with an uniform motion equal to that which p acquires at D. But Dg is greater than DG, (by the firſt axiom,) becauſe Dg is deſcribed by p with an accelerated motion, and DG is deſcribed in an equal time by an uniform motion equal to the motion of p at D. If the motion of p be perpetually retarded, then (by the fourth axiom) bD is greater than BD, and (by the third axiom) Dg is leſs than DG.



53. Let the motion of P in the line AV be uniform; and the motion of p in the line av be accelerated or retarded continually, (that is, let its velocity increaſe or decreaſe from one degree to another, by paſſing through all the intermediate degrees,) while P deſcribes any ſpaces BD and DG; let p deſcribe the ſpaces bd and dg; and, the motion of p being firſt accelerated, let bd be always leſs than BD, but dg greater than DG: then the velocity of p at d ſhall be equal to the conſtant velocity of P. For a greater ſpace than dg would be deſcribed by the

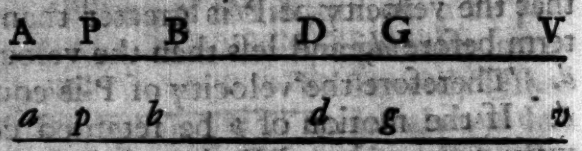
the point *p*, with its motion at *g* continued uniformly, in the ſame time P deſcribes DG, (by the ſecond axiom;) and therefore the velocity of P is leſs than the velocity of *p* at *g*. A leſs ſpace than *bd* would be deſcribed by the point *p*, with its motion at *b* continued uniformly, in the ſame time P deſcribes BD, (by the firſt axiom:) and therefore the velocity of P is greater than the velocity of *p* at *b*.

The motion of *p* is ſuppoſed to be accelerated in a continued manner; and therefore, the conſtant velocity of P being greater than the velocity of *p* at *b*, but leſs than the velocity of *p* at *g*, it muſt be equal to the velocity of *p* at ſome intermediate term of the ſpace *bg*. But it is demonſtrated, in the ſame manner, that the velocity of P is greater than the velocity of *p* at any term before *d*, and leſs than the velocity of *p* at any term after *d*. Therefore the velocity of P is equal to the velocity of *p* at *d*. If the motion of *p* be retarded continually, and *bd* be always greater than BD, but *dg* leſs than DG; it is demonſtrated, in the ſame manner, from the third and fourth axioms, that the conſtant velocity of P is equal to the velocity of *p* at *d*.

54. Let the motion of the point P be alſo accelerated; but, the motion of *p* being more accelerated, let *bd* be always leſs than BD, but *dg* greater than DG. In this caſe, the velocity of P at D is leſs than the velocity of *p* at *g*, becauſe a leſs ſpace than DG would be deſcribed by the former continued uniformly, and a greater ſpace than *dg* would be deſcribed by the latter continued uniformly, in the ſame time, (by the firſt and ſecond axioms.) The velocity of P at D is greater than the velocity of *p* at *b*, becauſe a greater ſpace than BD would be deſcribed by the former continued uniformly, and a leſs ſpace than *bd* (which is itſelf leſs than BD) would be deſcribed by the latter continued uniformly, in the ſame time, (by the ſame axioms.) Therefore, the motion of the point *p* being accelerated continually, the velocity of P at D muſt be equal to the velocity of *p* at ſome intermediate term of the ſpace *bg*. But, in the ſame manner as we have ſhewn, that it is greater than the velocity

of p at b, and leſs than the velocity of p at g, it is demonſtrated to be greater than the velocity of p at any term before p comes to d, and to be leſs than the velocity of p at any term after it paſſes d. Therefore the velocity of P at D is equal to the velocity of p at d.

55. Let the motion of the point p be now retarded continually, and the motion of P be alſo retarded; let bd be always greater than BD, and dg leſs than DG, (which happens when the motion of p is more retarded than the motion of P.) The velocity of P at D is greater than the velocity of p at g, becauſe a greater ſpace than DG would be deſcribed by the former continued uniformly, and a leſs ſpace than dg would be deſcribed by the latter continued uniformly, in the ſame time, (by the third and fourth axioms.) The velocity of P at D is leſs than the velocity of p at b, (by the ſame axioms.) Therefore, ſince the motion of p is ſuppoſed to decreaſe in a continued manner, the velocity of P at D muſt be equal to the velocity of p at ſome intermediate term of the ſpace bg. But, in the ſame manner as we have ſhewn, that the velocity of P at D is greater than the velocity of p at g, but leſs than the velocity of p at b, it is demonſtrated that the velocity of P at D is greater, or leſs, than the velocity of p at any term of the ſpace bg, d only excepted. Therefore the velocity of P at D is equal to the velocity of p at d.



56. When the motion of p is retarded continually, and the motion of P is accelerated continually; and bd is always greater than BD, but dg leſs than DG: then the velocity of p at b is greater than the velocity of P at B, becauſe a greater ſpace than bd would be deſcribed by the former continued uniformly, and a leſs ſpace than BD would be deſcribed by the latter continued uniformly, in the ſame time, (by the firſt and third axioms.) The velocity of p at g is leſs than the velocity of P at G, becauſe a leſs ſpace than dg would be deſcribed by the former continued uniformly, and a greater ſpace than DG would be deſcribed by the latter continued uniformly, in the ſame time,

(by

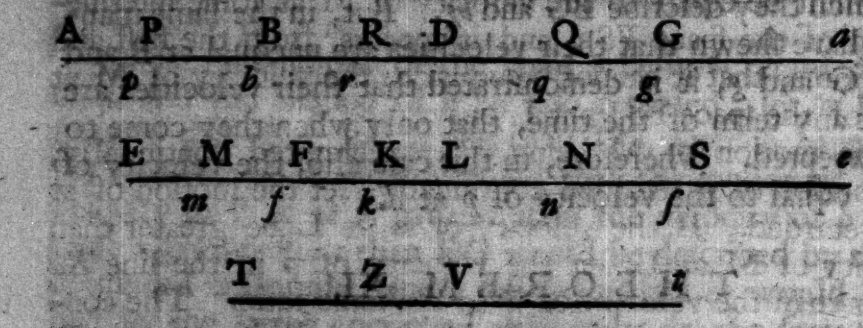
(by the ſecond and fourth axioms.) Therefore, ſince the motions of P and *p* increaſe and decreaſe in a continued manner, their velocities muſt be equal at ſome intermediate term of the time in which they deſcribe BG and *bg*. But, in the ſame manner as we have ſhewn that their velocities are unequal at B and *b*, and at G and *g*, it is demonſtrated that their velocities are unequal at any term of the time, that only when they come to D and *d* excepted. Therefore, in this caſe alſo, the velocity of P at D is equal to the velocity of *p* at *d*.

THEOREM XI.

57. *The motion of the point* P *upon the line* A*a being uniform, and the motion of the point p upon the ſame line being accelerated or retarded continually, let their velocities be equal at* D. *Then,* EM *being always determined from* AP *in any regular manner, and* E*m being determined from* A*p in the ſame manner; when* P *comes to* D, *let* M *come to* L *with a motion that is accelerated or retarded continually, and the velocity of m at* L *ſhall be equal to the velocity of* M *at* L.

In the firſt place, let the motion of M be continually accelerated, and the motion of *p* continually retarded. Let the points P, *p*, M and *m* deſcribe the ſpaces BD, *b*D, FL and *f*L reſpectively, in the time TV; and the ſpaces DG, D*g*, LS, L*ſ* in the time V*t*. Then (by art. 52.) *b*D is greater than BD, and D*g* leſs than DG. From which it follows, (art. 48.) that *f*L is greater than FL, and L*ſ* leſs than LS. In this caſe, the motion of *m* may be uniform, accelerated, or retarded. Let it firſt be uniform. By the firſt axiom, the point M, with its motion at F continued uniformly, would deſcribe in the time TV a ſpace leſs than FL, which is itſelf leſs than *f*L, the ſpace deſcribed in the ſame time by the point *m*; ſo that the velocity of M at F is leſs than the conſtant velocity of *m*. Therefore, ſince the motion of M is accelerated in a continued man-

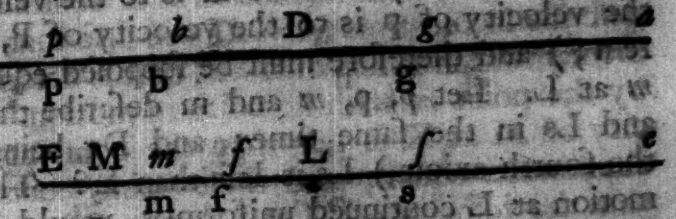
ner, by the ſuppoſition; if its motion at L be greater than the velocity of *m*, its velocity at ſome intermediate term of the ſpace FL, as K, muſt be equal to the conſtant velocity of *m*.



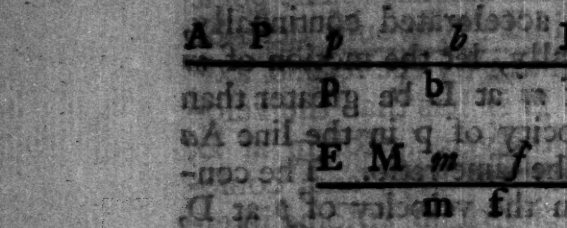
Let P, *p*, M and *m* deſcribe the ſpaces RD, *r*D, KL and *k*L in the ſame time ZV; and becauſe *r*D exceeds RD, (art. 52.) *k*L muſt be greater than KL. The point M, with its motion at K continued uniformly, would deſcribe in the time ZV a ſpace leſs than KL, (by the firſt axiom,) which is leſs than *k*L, that is deſcribed in the ſame time by the point *m* with its uniform motion. Therefore the velocity of M at K is leſs than the conſtant velocity of *m*. But they were equal; and theſe being contradictory, it appears, that the velocity of M at L is not greater than the conſtant velocity of *m*. The velocity of *m* is leſs than the velocity of M at S, becauſe L*ſ* is deſcribed by the former in the time V*t*, and a greater ſpace than LS (which exceeds L*ſ*) would be deſcribed in the ſame time by the latter continued uniformly, by the ſecond axiom. Therefore, if the velocity of *m* be greater than the velocity of M at L, it muſt be equal to the velocity of M at ſome intermediate term of the ſpace LS, as N. While M deſcribes LN, let P, *p* and *m* deſcribe DQ, D*q* and L*n*; and D*q* being leſs than DQ, (by art. 52.) L*n* is therefore leſs than LN. The point M, with its motion at N continued uniformly, would deſcribe a greater ſpace than LN (by the firſt axiom) while *m* with its uniform motion deſcribes L*n*; and therefore the velocity of *m* is leſs than the velocity of M at N. But they were equal; and theſe are contradictory. Therefore the velocity of *m* is neither greater nor leſs than the velocity of M at L, but preciſely equal to it. The demonſtration proceeds by the ſame ſteps when the velocity of M is accelerated, that of *p* retarded,

retarded, and the velocity of *m* is accelerated. For it appears, in the ſame manner, that the velocity of *m* at L is greater than the velocity of M at F, but leſs then the velocity of M at S; ſo that it muſt be equal to the velocity of M at ſome intermediate term of the ſpace FS: and it appears, from the firſt and ſecond axioms, that it cannot be equal to the velocity of M at any term of the ſpace FS, but L only.

58. The motion of M being ſtill accelerated continually, and the motion of *p* retarded continually, let the motion of *m* be alſo retarded. If the velocity of *m* at L be greater than the velocity of M at L, let the velocity of p in the line A*a* be greater than the velocity of P in the ſame ratio. The conſtant velocity of p being greater than the velocity of *p* at D, and the motion of *p* being retarded in a continued manner, the velocity of p ſhall be equal to the velocity of *p* at ſome term before it comes to D, as at *b*. Then, Em being determined from Ap in the ſame manner as EM is determined from AP, the motion of m ſhall be continually accelerated, (art. 48.) and the velocity of m at any term of the line E*e*, as L, ſhall be to the velocity of M at the ſame term, as the velocity of p is to the velocity of P, (by the tenth theorem,) or as the velocity of *m* at L is to the velocity of M at L. Therefore the velocity of m at L muſt be ſuppoſed equal to the velocity of *m* at L. Let *p*, p, *m* and m deſcribe the ſpaces *b*D, bD, *f*L and fL in the ſame time; and bD being greater than *b*D, (by the third axiom, the velocity of p being equal to the velocity of *p* at *b*,) fL muſt be greater than *f*L. The point m, with its motion at L continued uniformly, would deſcribe a greater ſpace than fL in that time, (by the ſecond axiom.) The point *m*, with its motion at L continued uniformly, would deſcribe a leſs ſpace than *f*L (which is leſs than fL) in the ſame time, (by the fourth axiom.) Therefore the velocity of m at L is greater than the velo-



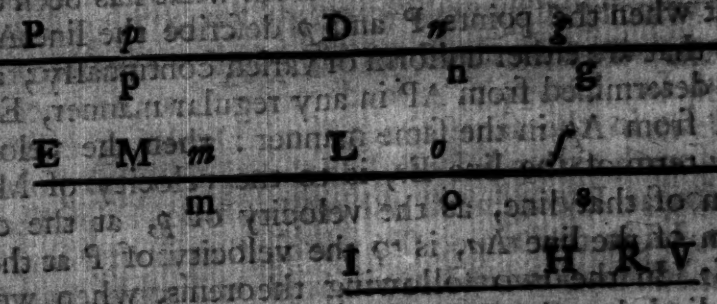
velocity of *m* at L. But theſe velocities were ſuppoſed equal; and theſe being contradictory, it follows, that the velocity of *m* at L is not greater than the velocity of M at L. If the velocity of *m* at L be leſs than the velocity of M at L, let the velocity of p be leſs than the velocity of P in the ſame ratio; and, ſince it is leſs than the velocity of *p* at D, let it be equal to the velocity of *p* at ſome ſubſequent term of the line A*a*, as at *g*. Em being determined from Ap in the ſame manner as EM is determined from AP, the velocity of m at L is to the velocity of M at L, as the velocity of p is to the velocity of P, (by the tenth theorem;) and therefore muſt be ſuppoſed equal to the velocity of *m* at L. Let *p*, p, *m* and m deſcribe the ſpaces D*g*, Dg, L*ſ* and Ls in the ſame time; and Dg being leſs than D*g*, (by the fourth axiom,) Ls is leſs than L*ſ*. The point m, with its motion at L continued uniformly, would deſcribe a ſpace leſs than Ls in that time, (by the firſt axiom;) and the point *m*, with its motion at L continued uniformly, would deſcribe a greater ſpace than L*ſ* (which exceeds Ls) in the ſame time, (by the third axiom.) Therefore the velocity of m at L is leſs than the velocity of *m* at L. But they were ſuppoſed equal; and theſe being contradictory, it appears, that the velocity of *m* at L is neither greater nor leſs than the velocity of M at L, but preciſely equal to it.



59. The motion of M being accelerated continually, as formerly, let the motion of *p* be alſo accelerated continually, and it is evident that the motion of *m* is alſo accelerated, (art. 14.) If the velocity of *m* at L be not equal to the velocity of M at L, let it firſt be greater, in the ſame ratio as IV is greater than IH; and (R being any point betwixt H and V) let the velocity of p be to the velocity of P (or the velocity of *p* at D) as IR is to IH; and, ſince it is greater than the velocity of *p* at D, ſuppoſe it equal to the velocity of *p* at *g*. Let p deſcribe

Dg

Dg with its uniform motion, in the ſame time that *p* deſcribes Dg, and Dg ſhall be greater than Dg, (by the ſecond axiom.) Let Em be determined from Ap in the ſame manner as EM is determined from AP; and, if m and *m* deſcribe the ſpaces Ls and L*ſ* in that time, Ls ſhall be greater than L*ſ*. The point *m*, with its motion at L continued uniformly, would deſcribe a leſs ſpace than L*ſ* in this time, (by the firſt axiom.) The point m, with its motion at s continued uniformly, would deſcribe a greater ſpace than Ls in the ſame time, (by the ſecond axiom.) Therefore the velocity of *m* at L is leſs than the velocity of m at s; but it is greater than the velocity of m at L, (in the ſame proportion as IV is greater than IR;) and, conſequently, it muſt be equal to the velocity of m at ſome intermediate term of the ſpace Ls, as o. In the ſame time that m deſcribes Lo, let *p*, p and *m* deſcribe the ſpaces D*n*, Dn and L*o* reſpectively.



The point *m*, with its motion at L continued uniformly, would deſcribe a ſpace leſs than L*o* in that time, (by the firſt axiom.) The point m, with its motion at o continued uniformly, would deſcribe a ſpace greater than L*o* in the ſame time, (by the ſecond axiom;) and, Dn being greater than D*n*, (becauſe the velocity of p always exceeds the velocity of *p* till *p* come to *g*,) Lo is greater than L*o*. Therefore the velocity of *m* at L is leſs than the velocity of m at o. But they were ſuppoſed equal; and theſe being contradictory, it appears, that the velocity of *m* at L is not greater than the velocity of M at L. In like manner it is demonſtrated, that the velocity of *m* at L is not leſs than the velocity of M at L; and therefore theſe velocities are equal to each other.

60. The other caſes of this theorem, when the motion of M is ſuppoſed to be retarded continually, are demonſtrated in the ſame

ſame manner: or they may be deduced from thoſe we have deſcribed, by the ninth theorem. When the motions begin or end at the terms D and L, the ſame demonſtration is applicable; ſince it is ſufficient, that the motions may be conceived to have begun before theſe terms, or to be continued after them. For the ſame reaſon, theſe demonſtrations may be applied, when L is a term where the motion of M ceaſes to be accelerated, being afterwards retarded; or where it ceaſes to be retarded, being afterwards accelerated. In like manner, the theorem may be extended to thoſe caſes, when the velocity of M is increaſed or diminiſhed at L by any finite or aſſignable quantity, by conceiving the velocity thus augmented or diminiſhed to have been produced by a continued acceleration or retardation while M came to L.

61. In general, it follows from what has been demonſtrated, that when the points P and *p* deſcribe the line A*a* with motions that are either uniform or varied continually; and, EM being determined from AP in any regular manner, E*m* is determined from A*p* in the ſame manner: then the velocity of *m*, at any term of the line E*v*, is to the velocity of M at the ſame term of that line, as the velocity of *p*, at the correſponding term of the line A*a*, is to the velocity of P at the ſame term.

62. In the two following theorems, when we ſay a ratio is a limit betwixt two other ratios, we mean no more, but that it is greater than the one, and leſs than the other.

THEOREM XII.

The velocity of a motion that is accelerated or retarded perpetually, is, at any term of the time, to the velocity of an uniform motion, in a ratio that is always a limit between the ratio of the ſpaces deſcribed by theſe motions in any equal times before that term, and the ratio of the ſpaces deſcribed by them in any equal times after it.

While the point P deſcribes the line A*a* with an uniform motion,

tion, let the point M deſcribe the line Ee with a motion that is accelerated or retarded perpetually. When P comes to D, let M come to L. Let BR and FK be ſpaces deſcribed by the points P and M in any time before they come to D and L; and let QG and NS be ſpaces deſcribed by them in any time after that term: and the velocity of M at L ſhall be to the conſtant velocity of P, in a ratio that is always a limit betwixt the ratio of FK to BR and the ratio of NS to QG.

First, let the motion of M be accelerated; and the point M, with its motion at N continued uniformly, would deſcribe a ſpace leſs than NS, in the ſame time the point P with its uniform motion deſcribes QG, (by the firſt axiom.) Therefore the velocity of the point M at N is to the conſtant velocity of the point P (art. 4.) in a leſs ratio than NS is to QG. But the motion of the point M being perpetually accelerated, its velocity at L is leſs than its velocity at N, and therefore is to the conſtant velocity of P in a leſs ratio than that of NS to QG. By the ſecond axiom, the point M, with its motion at K continued uniformly, would deſcribe a greater ſpace than FK, in the ſame time P with its uniform motion deſcribes the ſpace BR; and therefore the velocity of the point M at K, is to the conſtant velocity of the point P in a greater ratio than FK is to BR. But the velocity of the point M at L is greater than its velocity at K; and therefore is to the conſtant velocity of P in a greater ratio than FK is to BR. Thus it appears, that the velocity of the point M at L is to the conſtant velocity of P, in a ratio that is always a limit betwixt the ratio of NS to QG and the ratio of FK to BR; being in this caſe leſs than the former, and greater than the latter of thoſe ratios.

A P B R D Q G *a*

E M F K L N S *e*

63. If the motion of the point M be retarded perpetually, then (by the third axiom) the point M, with its motion at N continued uniformly, would deſcribe a greater ſpace than NS, in the ſame time P with its uniform motion deſcribes the ſpace QG; and therefore the velocity of M at N is to the con-

ſtant velocity of P in a greater ratio than NS is to QG. But the motion of M being now retarded, its velocity at L is greater than its velocity at N; and therefore its velocity at L is to the velocity of P in a greater ratio than NS is to QG. By the fourth axiom, the point M, with its motion at K continued uniformly, would deſcribe a ſpace leſs than FK, in the ſame time P deſcribes BR, with its uniform motion; and therefore the velocity of M at K is to the conſtant velocity of P in a leſs ratio than FK is to BR. But the velocity of M at L is leſs than its velocity at K; and, conſequently, the velocity of M at L is to the conſtant velocity of P in a leſs ratio than FK is to BR. Therefore the velocity of M at L is to the conſtant velocity of P, in a ratio that is always a limit betwixt the ratio of NS to QG and the ratio of FK to BR, being in this caſe greater than the former, and leſs than the latter of thoſe ratios. It is evident, that in either caſe the velocity of M at L is to the velocity of P, in a ratio that is always a limit betwixt the ratio of KL to RD, and that of LN to DQ; for this is only a particular caſe of the theorem.

A P B R D Q G *a*

E M F K L N S *e*

THEOREM XIII.

64. *The ſpace deſcribed by a motion that is accelerated or retarded perpetually, is to the ſpace deſcribed in the ſame time by an uniform motion, in a ratio that is a limit betwixt the ratio of the velocities of theſe motions at the beginning of the time and their ratio at the end of it.*

The points P and M being ſuppoſed to deſcribe the ſpaces DG and LS in the ſame time, and the motion of M being accelerated, as in the 62d article; then, ſince the point M, with its motion at L continued uniformly, would deſcribe a leſs ſpace than

than LS, in the time P deſcribes the ſpace DG with an uniform motion, (by the firſt axiom;) it follows, that the ſpace LS is to the ſpace DG in a greater ratio, than the velocity of M at L is to the conſtant velocity of P. The point M, with its motion at S continued uniformly, would deſcribe a greater ſpace than LS, in the time P deſcribes DG with its uniform motion, (by the ſecond axiom;) and, conſequently, the ſpace LS is to the ſpace DG in a leſs ratio, than the velocity of M at S is to the conſtant velocity of P. Therefore the ratio of the ſpace LS to the ſpace DG, is a limit betwixt the ratio of the velocity of M at L to the conſtant velocity of P, and the ratio of the velocity of M at S to the velocity of P; being greater than the former, and leſs than the latter of thoſe ratios.

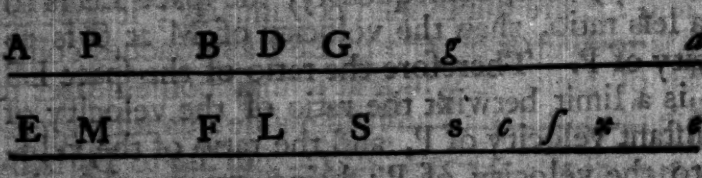
65. Let the motion of the point M be perpetually retarded, as in the 63d article; and, by the third axiom, the point M would deſcribe a greater ſpace than LS, with its motion at L continued uniformly, in the time P deſcribes DG; but the point M, with its motion at S continued uniformly, would deſcribe a leſs ſpace than LS in the time P deſcribes DG, (by the fourth axiom:) Therefore the ſpace LS is to the ſpace DG in a ratio that is leſs than the ratio of the velocity of M at L to the velocity of P, but greater than the ratio of the velocity of M at S to the velocity of P.

THEOREM XIV.

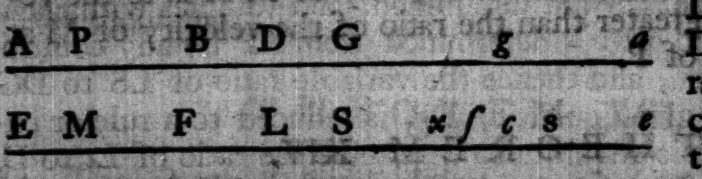
66. *The motion of the point* P *being uniform, but the motion of the point* M *continually varied, let the velocity of* P *be to the velocity of* M *at* L, *as a given line* Dg *is to* Lc; *let* Dg *be always to* L*ſ*, *as the ſpace* DG *deſcribed by* P *in any time, is to* LS *the ſpace deſcribed by* M *in the ſame time. Then, by diminiſhing the ſpaces* DG *and* LS *continually,* c*ſ may become leſs than any aſſignable magnitude.*

Let *cx* be any ſmall quantity aſſigned at pleaſure; and let it be added to Lc when the motion of M is accelerated, but ſub-

ducted from L*c* when the motion of M is retarded: and it is manifeſt, that, by diminiſhing LS, the velocity of M at S approaches continually to the velocity of M at L, ſo that their difference may become equal to the difference of any two unequal velocities that can be aſſigned, or leſs than it, how ſmall ſoever it may be. Let LS be diminiſhed till the difference of thoſe velocities be to the conſtant velocity of P as *cx* is to D*g*; and the velocity of M at S ſhall be to the velocity of P as L*x* is to D*g*.



When the motion of M is accelerated, it follows, from the 64th article, that the ratio of LS to DG, or of L*ſ* to D*g*, is greater than the ratio of the velocity of M at L to the velocity of P, or the ratio of L*c* to D*g*; but that the ſame ratio of LS to DG, or of L*ſ* to D*g*, is leſs than the ratio of the velocity of M at S to the velocity of P, or the ratio of L*x* to D*g*. Therefore L*ſ* is greater than L*c*, but leſs than L*x*; and, conſequently, *cſ* is leſs than *cx*. When the motion of M is retarded continually, then (by the 65th article) the ratio of LS to DG, or of L*ſ* to D*g*, is leſs than the ratio of the velocity of M at L to the velocity of P, or the ratio of L*c* to D*g*, but greater than the ratio of the velocity of M at S to the velocity of P, or the ratio of L*x* to D*g*. Therefore L*ſ* in this caſe is leſs than L*c*, but greater than L*x*; and, conſequently, *cſ* is leſs than *cx*. In the ſame manner it is demonſtrated, that if FL be always to BD as Ls is to D*g*, then, by diminiſhing the ſpaces FL, BD, which are deſcribed by M and P before they come to L and D, *cs* may become leſs than any given magnitude. And if FL, LS be ſpaces deſcribed by the point M in equal times, or in times that are to each other in any given proportion; and FL, LS, DG be to each other always in the ſame proportion as Ls, L*ſ* and D*g*: then, by diminiſhing the ſpaces FL and LS continually, *sſ* may become leſs



leſs than any given magnitude. It appears from what was ſhewn in the 44th article, that any motion muſt be ſuppoſed to be either uniform, or varied in a continued manner for ſome time, how ſmall ſoever that time may be; and therefore this theorem obtains univerſally.

67. Becauſe cf the difference betwixt Lf and Lc decreaſes ſo that it may become leſs than any given quantity, how ſmall ſoever, when DG and LS are diminiſhed continually; it appears, that the ratio of Dg to Lf (or of DG to LS) approaches continually to the ratio of Dg to Lc, ſo that it may come nearer to this ratio, than the ratio of Dg to any aſſignable quantity greater or leſs than Lc. For this reaſon, the ratio of Lc to Dg is by Sir ISAAC NEWTON called the *Limit* of the variable ratio of Lf to Dg, or of LS to DG, in a more reſtricted ſenſe of this term than that in which we made uſe of it in the twelfth and thirteenth theorems.

68. When the motion of the point M is continually accelerated from L to S, then Lf conſiſts always of two parts: the part Lc is invariable, and meaſures the velocity of M at L; the part cf is variable, and ariſes from the acceleration of the motion of M while it deſcribes LS. This latter part decreaſes continually when DG and LS are diminiſhed, and vaniſhes with them. Therefore, when EM is determined from AP by any conſtruction or equation, and thence the variable ratio of LS to DG, or of Lf to the given quantity Dg, is reduced to a rule or expreſſion, all that is requiſite to determine the ratio of Lc to Dg is, to diſtinguiſh betwixt Lc the invariable part of Lf and the variable part cf. And, for this purpoſe, it is ſufficient to ſuppoſe DG and LS to decreaſe, and to find what part of Lf continually decreaſes at the ſame time, and at length vaniſhes with LS; for this part is cf: which being rejected, the remainder Lc is to Dg as the velocity of M at L is to the conſtant velocity of P, or as the fluxion of EL is to the fluxion of AD. When the motion of M is continually retarded, then Lf is leſs than Lc by the difference cf which decreaſes and vaniſhes with LS, as before; and this part of the expreſſion of Lf being diſcovered and rejected in the ſame manner, the other part gives Lc, which is to the given line Dg as the fluxion of EL is to the fluxion of AD.

69. It

69. It is in this conciſe manner Sir ISAAC NEWTON moſt commonly determines the ratio of the fluxions of quantities. But we ſhall treat more fully of his method afterwards; and, ſince there have been various objections made againſt this doctrine, we ſhall demonſtrate its principal propoſitions immediately from the axioms. By tracing them to ſuch plain principles, their evidence may be more eaſily examined, and objections againſt them may either be obviated, or, if any doubt or difficulty remain, it may appear wherein preciſely it lyes. It is worth while to demonſtrate the chief propoſitions of this method in as clear and compleat a manner as poſſible, if by this means we can preſerve this ſcience from diſputes. Some of the preceeding theorems are ſo evident, that they are commonly admitted without a proof: but, becauſe we are delivering the elements of this doctrine, and have propoſed, in treating it, to imitate the ancient Geometricians, (who never increaſed the number of their principles without neceſſity,) we have deduced thoſe theorems from the axioms; that it might appear how few and plain the principles are which it is neceſſary for us to aſſume in demonſtrating it. It remains, before we proceed to enquire into the fluxions of particular quantities, that we ſhould ſay ſomething of the higher orders of fluxions.

70. When a motion is accelerated or retarded continually, the velocity may be itſelf conſidered as a variable or flowing quantity, and may be repreſented by a line that increaſes or decreaſes continually. When a velocity increaſes uniformly, ſo as to acquire equal increments in any equal times, its fluxion is meaſured by the increment which is generated in any given time. In this caſe, the velocity is repreſented by a line that is deſcribed with an uniform motion; and its fluxion, by the conſtant velocity of the point that deſcribes the line, or by the ſpace which this point deſcribes in a given time. When a velocity is not accelerated uniformly, but acquires increments in equal times that continually increaſe or decreaſe, then its fluxion at any term of the time is not meaſured by the increment which it actually acquires, but by that which it would have acquired if its acceleration had been continued uniformly from that term for a given time. And, in the ſame manner, when a motion is retarded con-

continually, the quantity by which it would be diminished in a given time, if its retardation was continued uniformly from any term, measures its fluxion at that term. While the point M describes the line EI, let the point Q describe the line IQ, so that IQ may be always equal to the space that would be described by the motion of M if it was continued uniformly for a given time. Then IQ shall always represent the velocity of M, and the velocity of the point Q shall represent the fluxion of the velocity of M;

E M
I Q

which therefore is measured, at any term of the time, by the space which would be described by Q with its motion at that term continued uniformly for a given time. The velocity of M is the fluxion of EM; and therefore the velocity of Q represents the fluxion of the fluxion of EM. Thus, when a fluxion of a quantity is variable, it may be considered itself as a fluent, and may have its fluxion, which is called the *Second fluxion* of that quantity. This may also have its fluxion, which is called the *Third fluxion* of the first fluent: and we shall shew afterwards, that motions may be easily conceived to vary in such a manner, as to give ground for admitting second fluxions, and those of any higher order.

71. The second fluxions are deduced from the first, in the same manner and upon the same principles as the first fluxions are deduced from their fluents: and therefore we shall subjoin in this place but one theorem concerning them.

LEMMA II.

When a motion is accelerated or retarded uniformly, the space described by it is an arithmetical mean betwixt the spaces that would be described in the same time by the motions at the beginning and end of that time continued uniformly.

Let

Let the point M deſcribe the line LS in any given time, with a motion that is accelerated or retarded uniformly; let LC and SH be the ſpaces that would be deſcribed in an equal time by its motions at L and S continued uniformly. Then the difference of SH and LS ſhall be equal to the difference of LS and LC.

Let the point *m* move from S to L, deſcribing always any ſpaces upon SL in times equal to thoſe in which they are deſcribed by M, but in a contrary order, (as in art. 47.) and the velocity of *m* at any term of the line LS, muſt be equal to the velocity of M when it comes to the ſame term of that line, by the ninth theorem. In the ſame time the point M deſcribes any ſpace L*z*, let *m* deſcribe S*x*; and, ſince the time in which M deſcribes *x*S is equal to the time in which *m* deſcribes it, (by the ſuppoſition,) it follows, that L*z* and *x*S are deſcribed by M in equal times. Therefore, ſince the motion of M increaſes or decreaſes uniformly, the difference of its velocities at L and *z* is equal to the difference of its velocities at *x* and S; and, conſequently, is equal to the difference of the velocities of *m* at *x* and S. From which it follows, that the ſum of the velocities of M and *m*, is always equal to the ſum of the velocities of M at L and of *m* at S, or to the ſum of the velocity of M at L added to its velocity at S. Therefore, by the eighth theorem, LS the ſpace deſcribed by M added to LS the ſpace deſcribed by *m* in the ſame time, is equal to the ſum of the ſpaces LC and SH that would be deſcribed in an equal time by the motions of M at L and S continued uniformly; and, conſequently, the difference of SH and LS is equal to the difference of LS and LC.

E L *z* R *x* C S H *s*
M *m*

72. Let LR and RS be any ſpaces deſcribed by the point M with a motion that is accelerated or retarded uniformly, in equal times that immediately ſucceed after one another; and, in the ſame time that M deſcribes the ſpace LS with this motion, it would deſcribe a ſpace equal to LS by its motion at R continued uniformly. For the velocity of M at R is an arithmetical mean betwixt its velocities at L and S, becauſe the motion of

of M increaſes or decreaſes uniformly; and therefore the point M, with its motion at R continued uniformly, would deſcribe a ſpace equal to half the ſum of LC and SH, in the ſame time that it would deſcribe LC with its motion at L, or SH with its motion at S continued uniformly. But, by this lemma, LS is equal to half the ſum of LC and SH; and the point M deſcribes LS with its accelerated motion in the ſame time that it would deſcribe LC with its motion at L continued uniformly. Therefore, in the ſame time that the point M deſcribes LS with a motion uniformly accelerated, it would deſcribe a ſpace equal to LS with its motion at R continued uniformly.

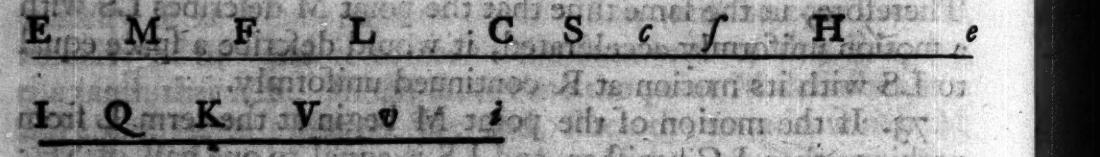
73. If the motion of the point M begin at the term L from nothing, then LC vaniſhes, and LS is equal to one half of SH; that is, when the motion begins from nothing, and is accelerated uniformly for any time, the ſpace deſcribed by it is one half of the ſpace deſcribed in an equal time by the motion that is acquired by this acceleration continued uniformly. This is one of the propoſitions diſcovered by GALILEUS; and ſeveral others of this kind may be demonſtrated in the ſame manner from the preceeding theorems, without having recourſe to the method of indiviſibles or of infiniteſimals.

THEOREM XV.

74. *Let the point* M *deſcribe the line* Ee *with any variable motion; and, in the ſame time that it would deſcribe* LC *with its motion at* L *continued uniformly, ſuppoſe that it would deſcribe* LS *if the acceleration or retardation of its motion was continued uniformly from that term. Then, if the velocity of* M *at* L, *or the firſt fluxion of* EL, *be repreſented by* LC, *the ſecond fluxion of* EL *may be meaſured by* 2CS.

The fluxion of the velocity of M, at any term of the time of its motion, is meaſured by the increment which it acquires in a given time when its acceleration is continued uniformly

from that term. The acceleration of the motion of M being ſuppoſed to be continued uniformly, till it deſcribe LS in the ſame time that it would deſcribe LC with its motion at L continued uniformly; let SH be to LC as the velocity which M would acquire in this manner at S is to its velocity at L: and, ſince the velocity of M at L is ſuppoſed to be repreſented by LC, the velocity which it would thus acquire at S will be re-



preſented by SH; and the increment of its velocity which would be generated if the acceleration of its motion was continued uniformly from L, in the ſame time that it would deſcribe LC with its motion continued uniformly from L, will be repreſented by the exceſs of SH above LC; which is equal to 2CS, by the laſt lemma. Therefore the fluxion of the velocity of M at L, or the ſecond fluxion of EL, is repreſented by 2CS. When the motion of M is retarded, then the decrement of the velocity of M that would be produced if the retardation of its motion was continued uniformly from L, in the ſame time in which M would deſcribe LC with its motion continued uniformly from L, is repreſented by the exceſs of LC above SH; and the ſecond fluxion of EL is repreſented by that exceſs, or by 2CS.

75. Or if we ſuppoſe, as in the 70th article, that the line IQ always repreſents the velocity of M, or the fluxion of EM, the velocity of the point Q will repreſent the fluxion of the velocity of M, or the ſecond fluxion of EM. Becauſe LC is ſuppoſed to repreſent the velocity of M at L, let IK be equal to LC, and Q ſhall come to K when M comes to L. The velocity of Q at K is meaſured by the ſpace which would be deſcribed by its motion at K continued uniformly, in the ſame time that M, with its motion at L continued uniformly, would deſcribe LC. Let KV be this ſpace; and, the motion of Q being ſuppoſed uniform while it deſcribes KV, the acceleration or retarda-

retardation of the motion of M is therefore continued uniformly for that time; and the velocity of M at S is to its velocity at L, as IV is to IK or LC. Therefore the point M would deſcribe ſpaces equal to IK and IV by its motions at L and S continued uniformly, in the ſame time that it deſcribes LS when the acceleration or retardation of its motion is continued uniformly; and, by the laſt lemma, KV is equal to 2CS. But KV repreſents the velocity of Q at K, or the ſecond fluxion of EL; which is therefore alſo repreſented by 2CS. When the motion of M is retarded, IQ decreaſes, the point Q moves from K towards I, and the ſecond fluxion of EM is in this caſe ſaid to be negative, being conſidered as a power that retards the generating motion, diminiſhing continually the firſt fluxion of EM. It appears from this theorem, that, as the firſt fluxion of a variable quantity, at any term of the time, is meaſured by the increment or decrement which would be produced if the generating motion was continued uniformly from that term for a given time; ſo its ſecond fluxion may be meaſured by twice the difference betwixt this increment or decrement, and that which would be produced if the acceleration or retardation of the generating motion was continued uniformly from that term for the ſame time.

76. Let K*v*, L*c* and L*f* be any other ſpaces that would be deſcribed in the ſame time by the uniform motion of Q, the motion of M at L continued uniformly, and the motion of M if its acceleration was continued uniformly from that term, reſpectively. Then the velocity of M acquired by this laſt motion at *f* ſhall be to its velocity at L, as I*v* is to IK or LC: and, the difference of thoſe velocities being to the velocity of M at L as 2*cf* is to L*c*, by the laſt lemma; it follows, that 2*cf* is to K*v* as L*c* is to LC. But K*v* is to KV as L*c* is to LC, (by the ſecond theorem;) and therefore *cf* is to CS in the duplicate ratio of L*c* to LC. It is alſo evident, that when LC and LS are ſuppoſed to decreaſe continually, the ratio of CS to LC decreaſes ſo that it may become leſs than any aſſignable ratio. For the ratio of the velocity of M at S to its velocity at L, or that of SH to LC, approaches continually to a ratio of equality, as the point S approaches to L, and may come nearer to it than

any assignable ratio of inequality. But CS is less than the difference of SH and LC; and therefore the ratio of CS to LC may become less than any assignable ratio. From this it follows, that when LC, which represents the first fluxion of EM, continually decreases, then 2CS, which represents its second fluxion, decreases so that its ratio to LC may become less than any assignable ratio.

77. Let FL and LS be spaces described by the point M in any equal times that succeed after one another; and let KV be described by Q in the same time M describes LS. When the motion of M is accelerated or retarded uniformly, its velocity at L, or the first fluxion of EL, may be measured by half the sum of FL and LS; and the second fluxion of EL may be measured by the difference of LS and FL. For the point M would describe a space equal to half the sum of FL and LS by its motion at L continued uniformly, in the same time that it describes LS with a motion accelerated or retarded uniformly, by the 72d article; and KV, or 2CS, (which measures the second fluxion of EL when LC measures its first fluxion,) is equal to the difference of the spaces LS and FL. When the motion of M is accelerated uniformly, the space LS which is described by M in a given time, is equal to LC, that represents the first fluxion of EL, added to the half of KV that represents the second fluxion of EL; and LS is equal to the difference of LC and one half of KV when the motion of M is retarded uniformly. In other cases, when the acceleration or retardation of the motion of M is not uniform, but increases or decreases continually while it describes the space FS, LC is not equal to half the sum of FL and LS; but it follows from the 66th and 67th articles, that its ratio to half their sum approaches continually to a ratio of equality as its limit, when those spaces are continually diminished. And in the same manner it appears, that, by diminishing LC, the ratio of KV to the difference of LS and FL, and the ratio of LS to the sum or difference of LC and one half of KV, continually approach to a ratio of equality, so that they may come nearer to it than any assignable ratio of inequality. The ratio of *of* to CS (the differences betwixt the spaces described by M and those which would be described in the same times by its motion conti-

continued uniformly from L) approaches continually to the duplicate ratio of Lc to LC, by diminiſhing thoſe ſpaces, ſome caſes excepted that will be deſcribed afterwards. There are theorems analogous to theſe which relate to the higher orders of fluxions; but we ſhall demonſtrate them afterwards, and proceed now to enquire into the fluxions of geometrical magnitudes. Beſides the preceeding general theorems, there are others concerning the compoſition and reſolution of motion, which are ſometimes conſidered as the grounds of this method; but they may rather ſerve for applying this general doctrine to particular caſes; and therefore we refer them to another place.

CHAP. II.

Of the Fluxions of plane rectilineal Figures.

PROPOSITION I.

78. *THE fluxion of a parallelogram of an invariable altitude, is always meaſured by a parallelogram of the ſame altitude deſcribed upon the right line which meaſures the fluxion of the baſe.* FIG. 16. Pl. 3.

While the point P deſcribes the baſe AO, let the given right line PM, by moving parallel to itſelf, generate the parallelogram APMF. When P comes to D, let PM come to DE; and, if the fluxion of the baſe at that term of the time be repreſented by DG, the fluxion of the parallelogram at the ſame term ſhall be repreſented by the parallelogram EG.

When the motion of P is uniform, (that is, when it deſcribes equal ſpaces in any equal times,) the right line PM deſcribes equal parallelograms in equal times, (Elem. 36. 1.) Therefore the motion of PM is alſo uniform, (art. 14.) and in the ſame time that the point P deſcribes DG with an uniform motion, the right line PM deſcribes the parallelogram EG with an uniform motion. Therefore, the fluxion of the baſe AD being repreſented.

ſented by DG, the fluxion of the parallelogram AE is repreſented by the parallelogram EG, by art. 11.

79. Since P is a point or term of the line PM, which is of a given or invariable magnitude, and is ſuppoſed to move always parallel to itſelf, any point in this line moves with the ſame velocity as P. According as the motion of P is accelerated or retarded, the motion of the right line PM is accelerated or retarded; and, in the ſame time that the point P would deſcribe DG with its motion at D continued uniformly, the right line PM would deſcribe the parallelogram EG with its motion continued uniformly from the ſame term. If this can be ſuppoſed to need any other proof, it may be demonſtrated from the axioms in the following manner.

80. When the motion of P is continually accelerated, the motion of PM is alſo accelerated continually, (Elem. 1. 6. and art. 14.) Let the point *p* deſcribe the baſe with an uniform motion equal to that of P at D; let *pm*, equal and parallel to PM, generate the parallelogram A*pm*F: and the conſtant velocity of *pm* ſhall be equal to the velocity of PM at the term or moment when P comes to D. For, while *p* with its uniform motion deſcribes any ſpaces *g*D and DG, let P with its accelerated motion deſcribe the ſpaces *k*D and DK; and, ſince the velocity of P at D is equal to the conſtant velocity of *p*, DK is greater than DG, (by ax. 1.) and D*g* greater than D*k*, (by ax. 2.) Complete the parallelograms EG, E*g*, EK, E*k*; and EK ſhall be greater than EG, but E*k* leſs than E*g*. By ax. 2. the right line PM would deſcribe a greater ſpace than EK, with its motion at the term K continued uniformly, in the ſame time P deſcribes DK, or *pm* deſcribes EG. Therefore the velocity of PM at K is greater than the conſtant velocity of *pm*. By ax. 1. the right line PM, with its motion at *k* continued uniformly, would deſcribe a leſs ſpace than E*k* in the time P and *p* deſcribe *k*D and *g*D; and in the ſame time *pm* with a conſtant velocity deſcribes E*g*, which is greater than E*k*. Therefore the velocity of PM at *k* is leſs than the conſtant velocity of *pm*. In the ſame manner it is demonſtrated, that the velocity of *pm* is leſs that the velocity of PM at any term after P paſſes D, but is greater than the velocity of PM at any term before

P

P comes to D. Therefore the velocity of *pm* is equal to the velocity of PM at the term or moment when P comes to D. When the motion of P is retarded, it appears in the ſame manner, from ax. 3. & 4. that the motion of PM at D is equal to the conſtant velocity of *pm*. The uniform velocity of *p*, or the motion of P at D, being meaſured by DG, the motion of *pm* is meaſured by the parallelogram EG, (by art. 78.) which therefore meaſures the motion of PM at D, or the fluxion of the parallelogram AM when AM becomes equal to AD. In the ſame manner it is demonſtrated in general, that when a given line, by revolving about a given center or axis, deſcribes any area; or when a given ſurface, by moving parallel to itſelf, or by revolving on a given axis, generates a ſolid: the motion with which the area or ſolid flows at any given term is always the ſame, when the velocity of the generating figure at that term is the ſame, whatever variation the motion of the generating figure may be ſubject to before or after that term.

81. Let the right lines AO, AV be given in poſition; and, FIG. 17. while the point P deſcribes the baſe AO, let the right line PM, by moving parallel to itſelf, generate the triangle APM. At the ſame time, let the point *p* deſcribe the right line *ao*; and a given or invariable right line *pm*, by moving parallel to itſelf, generate a parallelogram *am* always equal to the triangle APM. When P comes to D, let *p* come to *d*, PM to DE, and *pm* to *de*: Then the velocity with which the baſe AD flows, is the ſame as the velocity of the point P at the term or moment when it comes to D, (art. 10.) and the velocity with which the triangle ADE flows at that term, is the ſame as the velocity of the invariable right line *pm* when *p* comes to *d*. Therefore, when the velocity of P is given at any term of the time, to determine thence the velocity of the right line *pm* at that term, is the ſame as from the fluxion of the baſe AP to determine the fluxion of the triangle APM, (art. 11.) and if the velocity of *pm* is given, to determine thence the velocity of P, is the ſame as from the fluxion of the triangle APM to determine the fluxion of the baſe AP.

82. It is manifeſt, that when the place of the point P at any term of the time is given, the place of the right line *pm* at that

term

term is found by applying upon the given right line *af* a parallelogram *am* equal to the triangle APM, (the angle *mpo* being always supposed equal to MPO.) If the place of *pm* is given, and that of P is required; let AO be of such a magnitude that OV parallel to PM may be equal to 2*af* or 2*pm*; and take AP a mean proportional betwixt *ap* and AO. For if *ap* be to AP as AP is to AO, or as PM is to OV, (which is equal to 2*af*;) the rectangle contained by *ap* and *af* shall be equal to half the rectangle contained by AP and PM, and the parallelogram *am* shall be equal to the triangle APM.

LEMMA III.

83. *When the base increases uniformly, the triangle increases with a motion that is perpetually accelerated; but when the base decreases uniformly, the triangle decreases with a motion that is perpetually retarded.*

The same things being supposed as in the two last articles, the motion of the point P is the motion with which the base flows; and the motion of the given right line *pm* is the same as that with which the triangle flows. Let the right line *pm* describe the parallelograms *be* and *dh* in any equal times that immediately succeed after one another; and let the point P describe the right lines BD and DG in the same equal times. Let BC, DE and GH, parallel to PM, meet AP in C, E and H. Then, because the spaces described by *pm* are supposed to be always equal to the spaces described in the same time by PM, the parallelogram *be* will be equal to the trapezium BDEC, and the parallelogram *dh* equal to the trapezium DGHE. Because the motion of P is uniform, BD is equal to DG, and the trapezium DGHE is greater than BDEC in the same proportion as the sum of DE and GH is greater than the sum of DE and BC, or the sum of AG and AD is greater than the sum of AD and AB; and the parallelogram *dh* is greater than *be* in the same proportion. Therefore, when the base increases uniformly, the spaces *be* and

and *dh* described by *pm* in any equal times that succeed after one another, perpetually increase; and the motion of *pm* is accelerated perpetually. But when the base decreases, the point P moves from O towards A; the right line *pm*, by moving from *o* towards *a*, describes in any equal times that succeed after one another the parallelograms *hd* and *eb* that perpetually decrease; and its motion in this case is perpetually retarded. The motion with which the triangle flows, is measured by the motion of *pm*; and is therefore perpetually accelerated when the base increases uniformly, but perpetually retarded when the base decreases uniformly.

84. When the triangle APM increases uniformly, the base increases with a motion that is perpetually retarded; but when the triangle decreases uniformly, the base decreases with a motion that is perpetually accelerated. For, when the triangle APM increases or decreases uniformly, the motion of the right line *pm* is uniform, by the supposition. While *pm* describes the equal parallelograms *be* and *dh* in any equal times, let the point P describe the right lines BD and DG; and, the trapezium BDEC being equal to the parallelogram *be*, and DGHE equal to the parallelogram *dh*, the trapezium BDEC is equal to the trapezium DGHE, and BD is greater than DG in the same proportion as the sum of DE and GH is greater than the sum of DE and BC. Therefore, when the triangle increases uniformly, or the motion of *pm* from *a* towards *o* is uniform, the spaces BD and DG described by the point P in any equal times perpetually decrease, and its motion is perpetually retarded. But, when the triangle decreases uniformly, or the motion of *pm* from *o* towards *a* is uniform, the spaces GD and DB described by P in any equal times perpetually increase, and its motion is accelerated perpetually. FIG. 18.

85. All the rules for the operations in the direct method of fluxions may be deduced from the two following propositions; and there can hardly remain any ground for objections against it when these are established in an unexceptionable manner. We shall therefore demonstrate them at some length, by the method which seems to set the evidence of this doctrine in the clearest light, and to resolve in the most satisfying manner the difficulties that have been raised against its truth or accuracy.

PROP. II.

FIG. 17. & 19. *The ſides* AD, AE *of the triangle* ADE *being given in poſition, and the angle* ADE *being alſo given; in the ſame time that the motion with which the baſe* AD *flows, continued uniformly, would generate any right line* DG, *the motion with which the triangle* ADE *flows, continued uniformly, would generate the parallelogram* EG. *Or: The fluxion of the baſe* AD *being repreſented by* DG, *the fluxion of the triangle* ADE *is accurately meaſured by the parallelogram* EG.

While the point P deſcribes the baſe AO, and the variable right line PM by moving parallel to itſelf generates the triangle APM, let the invariable right line *pm*, by moving parallel to itſelf along the right line *ao*, generate the parallelogram *am* always equal to the triangle APM, ſo that the ſpaces deſcribed by *pm* may be always equal to thoſe deſcribed in the ſame time by PM, as in the preceeding articles: Then the motion with which the baſe flows, or its fluxion, ſhall be always meaſured by the velocity of the point P; and the motion with which the triangle APM flows, or its fluxion, ſhall be always meaſured by the velocity of the invariable line *pm*. When AP becomes equal to AD, let *ap* become equal to *ad*, (that is, let *p* come to *d* when P comes to D;) and the right lines PM, *pm* ſhall come to DE and *de* at the ſame term of the time. Suppoſe that, if the motion of P was continued uniformly from this term, it would deſcribe the line DG in any given time; and that, if the motion of *pm* was continued uniformly from the ſame term, it would deſcribe a ſpace equal to the parallelogram *ek*, in the ſame given time: Then ſhall the parallelogram *ek* be equal to the parallelogram EG.

FIG. 17. *Caſe* 1. When the baſe increaſes uniformly, or the motion of the point P from A towards O is uniform; the motion of *pm* from

from *a* towards *o* is a motion perpetually accelerated, by the laſt lemma. Let the point P deſcribe BD and DG in equal times that ſucceed immediately after each other; and let *pm* deſcribe the parallelograms *be* and *dh* in the ſame equal times. Let BC, DE and GH parallel to PM meet AV in C, E and H; and the parallelogram *be* ſhall be equal to the trapezium BDEC, and *dh* equal to DGHE, by the ſuppoſition. The motion of *pm* being accelerated perpetually, it follows from ax. 1. that the parallelogram *ek* is leſs than the parallelogram *dh*; becauſe the ſpace *dh* is deſcribed by *pm* with an accelerated motion, and *ek* is the ſpace that would be deſcribed in the ſame time by *pm* with its motion continued uniformly from the beginning of that time without any acceleration. By ax. 2. the ſame parallelogram *ek* is greater than *be*, which was deſcribed by *pm* in an equal time before its velocity at the term when it comes to *d* was acquired. Therefore the parallelogram *ek* is leſs than the trapezium DGHE, but greater than the trapezium BDEC. It is evident alſo, that the parallelogram EG is leſs than DGHE, but greater than BDEC. I ſay further, that the parallelogram *ek* is preciſely equal to EG. For, if it is not equal to EG, it muſt be greater or leſs than it. Let *ek* firſt be greater than EG, and produce DE beyond E to R, till DR be greater than DE in the ſame ratio; and, completing the parallelogram DRLG, it ſhall be to EG as DR is to DE, (Elem. 1. 6.) or as *ek* is to EG, (by the ſuppoſition:) and therefore the parallelogram RG will be equal to *ek*. Let RL meet CH in N, and NQ parallel to DE meet the baſe in Q. Suppoſe that *pm* would deſcribe the parallelogram *ex*, by its motion continued uniformly from the term when *p* comes to *d*, in the ſame time P deſcribes DQ with its uniform motion. Then, the ſpaces deſcribed by any uniform motion being in the ſame proportion as the times in which they are deſcribed, (by theor. 1. art. 16.) the parallelogram *ek*, or RG, ſhall be to the parallelogram *ex* as DG is to DQ, or as RG is to RQ; and therefore the parallelograms *ex* and RQ are equal. But, while the point P deſcribes DQ, the right line *pm* deſcribes a ſpace equal to the trapezium DENQ, by the ſuppoſition; and, its motion being perpetually accelerated during this time, it follows from ax. 1. that DENQ is greater

than *ex* the ſpace which would have been deſcribed in the ſame time by *pm* if its motion had been continued uniformly from the beginning of that time without any acceleration. And, ſince DR is greater than DE in the ſame proportion as *ek* is ſuppoſed greater than EG, the parallelogram RQ is greater than the trapezium DENQ; and therefore is ſurely greater than *ex*. But RQ was proved equal to *ex*: And theſe being contradictory, it follows, that the parallelogram *ek* is not greater than the parallelogram EG.

86. Let us ſuppoſe now that the parallelogram *ek* is leſs than the parallelogram EG; and, D*r* being ſuppoſed leſs than DE in the ſame proportion, complete the parallelogram *r*G; and, *r*G being to EG as D*r* is to DE, (Elem. 1. 6.) or as *ek* is to EG, *r*G muſt be equal to *ek*. Let *rl* produced meet CE in *n*, and *nq* parallel to DE meet the baſe in *q*. Suppoſe that *pm* would deſcribe the parallelogram *ex*, by its motion continued uniformly from the term when *p* comes to *d*, in a time equal to that in which P deſcribes *q*D. Then, by theor. 1. the parallelogram *ek*, or *r*G, ſhall be to the parallelogram *ex* as DG is to *q*D, or as the parallelogram *r*G is to *rq*; and therefore *ex* is equal to *rq*. But, while P deſcribes *q*D, the right line *pm* deſcribes a ſpace equal to the trapezium *qn*ED, by the ſuppoſition; and, its motion being perpetually accelerated during this time, it follows from ax. 2. that the trapezium *qn*ED is leſs than *ex* the ſpace which would be deſcribed in an equal time by *pm* with the motion continued uniformly which it acquires at the term when *p* comes to *d*. And, the trapezium *qn*ED being greater than the parallelogram *rq*, (ſince D*r* is leſs than DE in the ſame proportion as *ek* is ſuppoſed leſs than EG,) it follows, that the parallelogram *ex* is greater than the parallelogram *rq*. But *ex* was found equal to *rq*: And theſe being contradictory, it follows, that the parallelogram *ek* is not leſs than the parallelogram EG. Nor is *ek* greater than EG; and, conſequently, theſe parallelograms are equal to each other. Therefore, when P and *p* come to D and *d*, if the motion of *pm* was continued uniformly from that term, it would deſcribe a ſpace equal to the parallelogram EG, in the ſame time that the point P deſcribes DG with its uniform motion; and, the fluxion of the baſe AD being

being repreſented by DG, the fluxion of the triangle ADE (which is meaſured by the velocity of *pm* at the term when P comes to D and *p* to *d*) is repreſented by the parallelogram EG.

87. *Caſe* 2. Let the baſe decreaſe uniformly, or the motion of P be uniform from O towards A; and the motion of the right line *pm* from *o* towards *a* ſhall be perpetually retarded, by lemma 3. In this caſe, (the conſtruction and figure being the ſame as in the former,) the ſpace *ek* is greater than the parallelogram *eb*, or the trapezium DECB, by ax. 3. and *ek* is leſs than the parallelogram *hd*, or the trapezium GHED, by ax. 4. I ſay further, that the parallelogram *ek* is preciſely equal to the parallelogram EG. For, if *ek* be not equal to EG, let it firſt be greater than EG; and, DR being ſuppoſed to be greater than DE in the ſame proportion, let RL parallel to the baſe meet GH in L: and *ek* will be equal to RG. Then, in the ſame manner as in the 85th article, the parallelogram RG, or *ek*, is to the parallelogram RQ as the baſe DG is to DQ: and it follows from theor. 1. that, in the time P deſcribes QD, *pm* would deſcribe a ſpace equal to the parallelogram RQ by its motion continued uniformly from the term when P comes to D; and, by ax. 4. this ſpace muſt be leſs than the trapezium QNED, which is equal to the ſpace that was deſcribed by *pm* with its retarded motion before *p* came to *d* while P deſcribed QD. But the parallelogram RQ is greater than the trapezium QNED, ſince DR is greater than DE in the ſame proportion as the parallelogram *ek* is ſuppoſed greater than EG: And theſe being contradictory, it follows, that the parallelogram *ek* is not greater than the parallelogram EG. Let *ek* therefore be ſuppoſed leſs than EG, and Dr leſs than DE in the ſame proportion; then, completing the parallelogram *r*G, as in the laſt article, *r*G ſhall be equal to *ek*: and, ſince the parallelogram *r*G is to *rq* as the baſe DG is to D*q*, it follows from theor. 1. that, in the ſame time P deſcribes D*q*, the right line *pm* would deſcribe a ſpace equal to the parallelogram *rq*, by its motion continued uniformly from the term when *p* comes to *d*. But this ſpace (by ax. 3.) muſt be greater than the trapezium DE*nq*, which is equal to the ſpace deſcribed in the ſame time by *pm* when its motion is perpetually retarded from the

ſame

ſame term: and the parallelogram *rq* is leſs than the trapezium DE*nq*, ſince D*r* is leſs than DE in the ſame proportion as the parallelogram *ek* is ſuppoſed to be leſs than EG. But theſe are contradictory; and therefore *ek* is not leſs than EG. Nor is *ek* greater than EG; and therefore *ek* and EG are equal.

FIG. 19. 88. *Caſe* 3. Let the triangle APM increaſe uniformly, or the motion of *pm* from *a* towards *o* be uniform; and the motion of P from A towards O ſhall be perpetually retarded, by art. 84. In this caſe, if *ek* be ſuppoſed greater than EG, let DR be greater than DE in the ſame proportion; and the parallelogram RG ſhall be equal to *ek*. Let RL parallel to the baſe meet EH in N, and NQ parallel to DE meet the baſe in Q. Then, ſince the parallelogram RG is to RQ as DG is to DQ; and it is ſuppoſed, that, in the time *pm* deſcribes the parallelogram *ek* (or RG) with its uniform motion, the point P would deſcribe DG with its motion continued uniformly from the term when it comes to D: it follows, from theor. 1. that the point P would deſcribe DQ, by the ſame motion continued uniformly, in the time *pm* deſcribes a ſpace equal to RQ. Therefore the point P would deſcribe a line leſs than DQ, by the ſame motion continued uniformly, in the time *pm* deſcribes with its uniform motion a ſpace equal to the trapezium DENQ, which is leſs than RQ. But, while *pm* deſcribes a ſpace equal to DENQ, the point P deſcribes DQ with a motion perpetually retarded, (by art. 84.) and therefore, by ax. 3. it would deſcribe a line greater than DQ in this time, by its motion continued uniformly from the term when it comes to D: And theſe being contradictory, it appears that *ek* is not greater than EG. Let *ek* be now leſs than EG; and, if D*r* be leſs than DE in the ſame proportion, *ek* ſhall be equal to *r*G. Let *rl* parallel to the baſe meet CH in *n*, and *nq* parallel to DE meet the baſe in *q*. Then, ſince the parallelogram *r*G is to *rq* as the baſe DG is to D*q*, it follows, from theor. 1. that, in the ſame time *pm* deſcribes a ſpace equal to *rq* with its uniform motion, the point P would deſcribe a line equal to D*q* by its motion continued uniformly from the term when it comes to D; and therefore P would deſcribe a greater line than D*q*, by the ſame motion continued uniformly, in the time *pm* deſcribes a ſpace equal to the

trape-

Fig. 13.

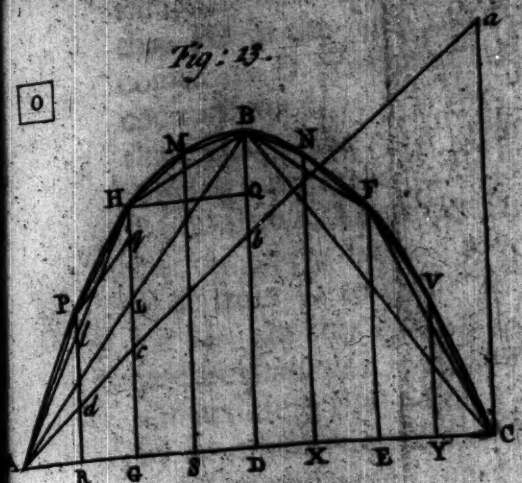


Fig. 15.

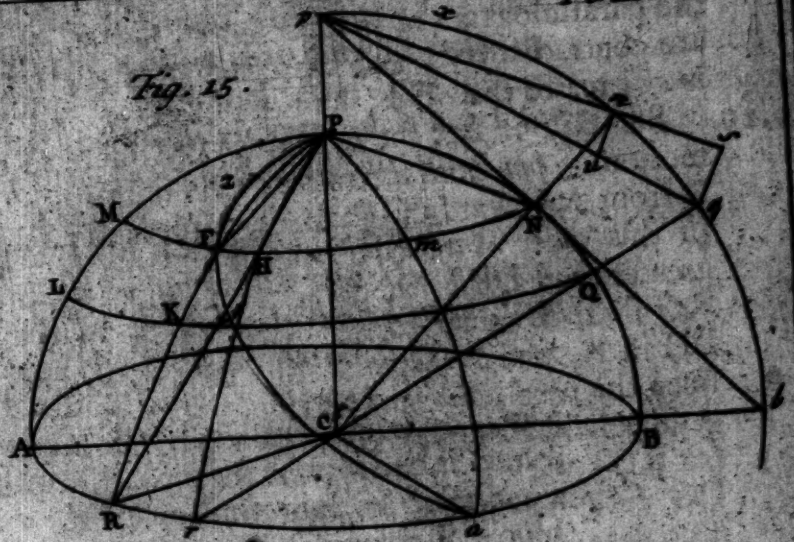


Fig. 14.

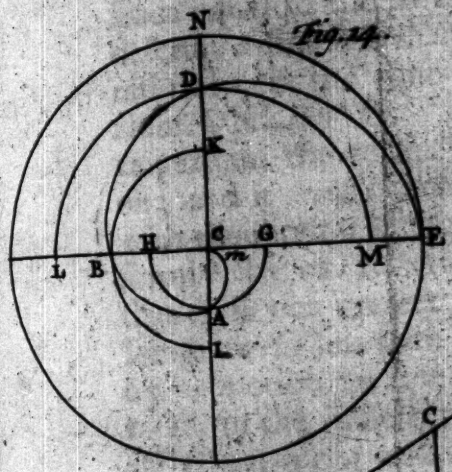
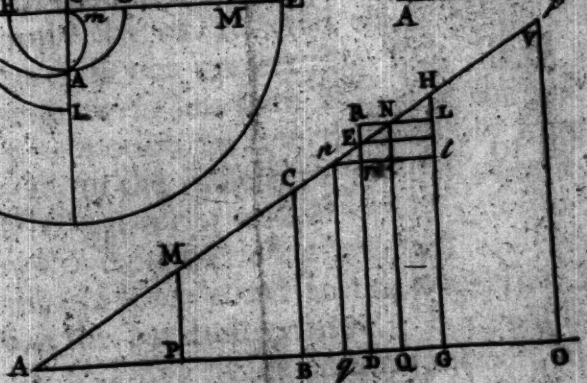


Fig. 16.



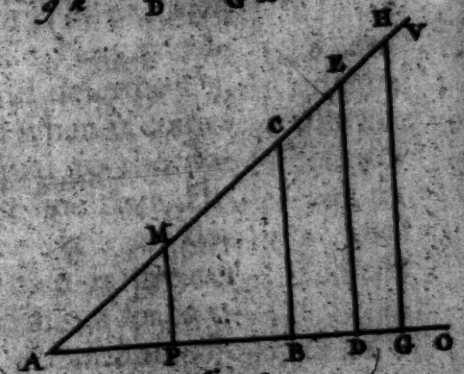


Fig. 17.

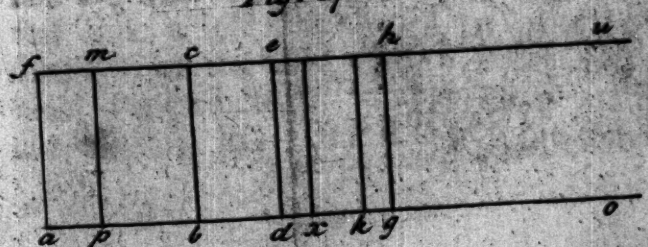
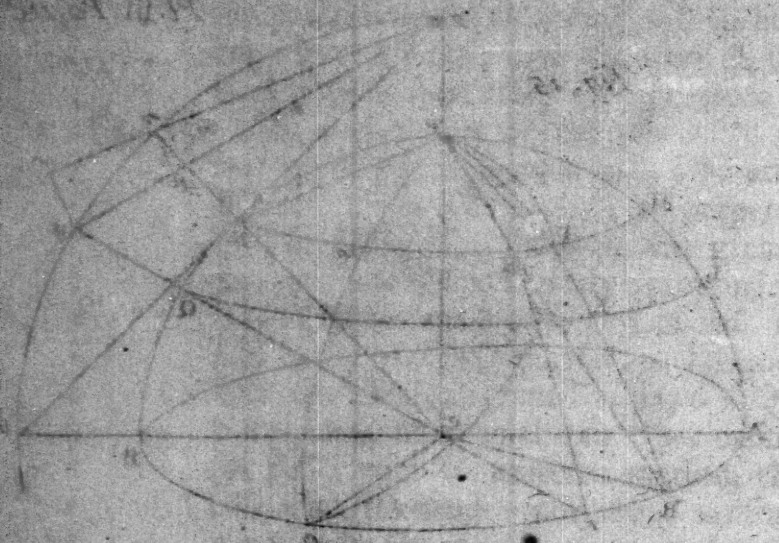
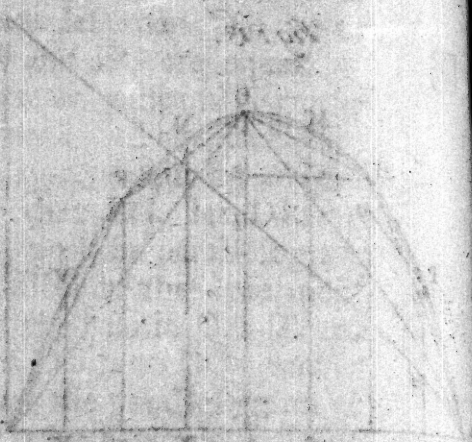
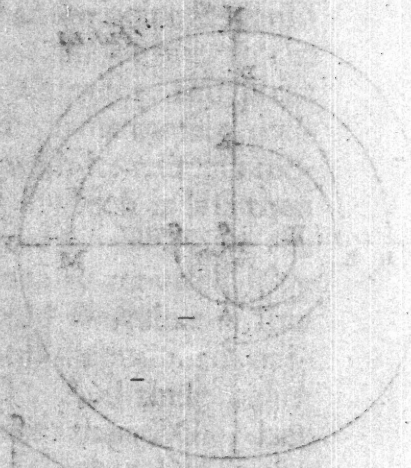
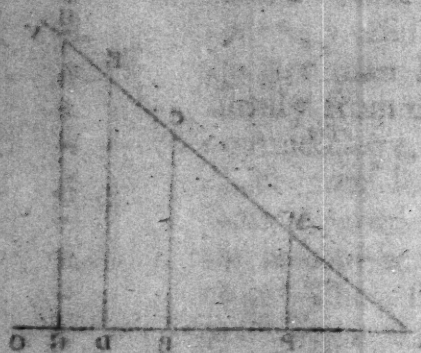


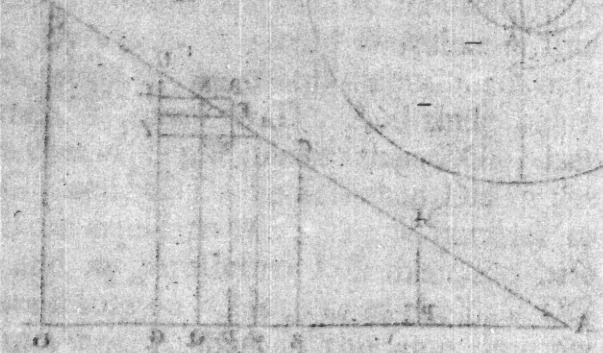
Fig. 18.

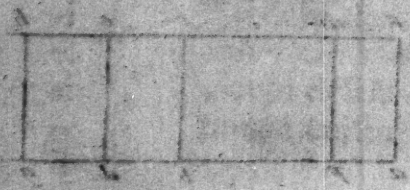


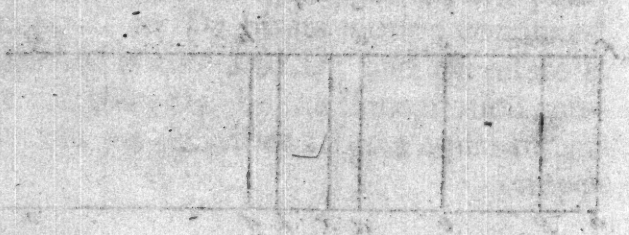












trapezium *qn*ED, which is greater than the parallelogram *rq*. But in this time the point P deſcribes *q*D with a retarded motion, (by art. 84.) and therefore it would deſcribe a leſs line than *q*D in the ſame time with the motion that remains when it comes to D continued uniformly, by ax. 4. And theſe being contradictory, it follows, that the parallelogram *ek* is not leſs than EG. Nor is *ek* greater than EG; and therefore theſe parallelograms are equal to each other.

89. *Caſe* 4. Let the triangle APM decreaſe uniformly, or the motion of *pm* from *o* towards *a* be uniform; and the motion of P from O towards A ſhall be perpetually accelerated, by art 84. In this caſe, if *ek* was equal to any parallelogram RG greater than EG, (the conſtruction being ſimilar to that of the laſt article,) the point P would deſcribe a line equal to DQ, by its motion continued uniformly from the term when it comes to D, in the ſame time *pm* with its uniform motion deſcribes a ſpace equal to the parallelogram RQ; and the point P would deſcribe a line leſs than DQ, by the ſame motion continued uniformly, in the time *pm* deſcribes a ſpace equal to the trapezium QNED. But, while *pm* deſcribes a ſpace equal to this trapezium, the point P deſcribes QD with an accelerated motion; and it would deſcribe a line greater than QD in this time, with the motion it acquires when it comes to D continued uniformly, by ax. 1. And theſe being contradictory, it follows that *ek* is not greater than EG. If *ek* was equal to any parallelogram *r*G leſs than EG, then the point P would deſcribe D*q*, by its motion continued uniformly from the term when it comes to D, in the ſame time *pm* deſcribes a ſpace equal to the parallelogram *rq*; and P would deſcribe a greater line than D*q*, by the ſame motion continued uniformly, in the time *pm* deſcribes a ſpace equal to the trapezium DE*nq*, which is greater than the parallelogram *rq*. But, while *pm* deſcribes a ſpace equal to DE*nq* by its uniform motion, the point *p* deſcribes the line D*q* with an accelerated motion; and it would deſcribe in this time a leſs line than D*q*, with its motion continued uniformly from the term when it comes to D, by ax. 1. And theſe being contradictory, it appears, that the parallelogram *ek* is not leſs than EG. But it is not greater than EG; and therefore

fore theſe parallelograms are equal to each other, when either the baſe AD, or the triangle ADE, increaſe or decreaſe uniformly.

90. The laſt three caſes might have been demonſtrated from the firſt, by art. 47. & 57. But, for the illuſtration of this method, we choſe to deduce them immediately from the axioms and the firſt theorem. When the motion of *pm* is accelerated, and the motion of P accelerated or retarded, the propoſition may be demonſtrated in the ſame manner. But all the other caſes of this propoſition are briefly deduced from the firſt, by the eleventh theorem, thus. In general, let the motion of P, while it deſcribes AO, be accelerated or retarded at pleaſure; but let it come to D with a motion, that, continued uniformly for any given time, would generate the line DG: and the right line *pm* ſhall come to *de* with a motion which, if it was continued uniformly for the ſame time, would generate a parallelogram *ek*
FIG. 20. equal to the parallelogram EG. For, ſuppoſe QN parallel to PM to generate the triangle AQN; let the motion of Q be uniform, and equal to that with which P comes to D; let *qn*, equal and parallel to *pm*, generate the parallelogram *aqnf*, always equal to the triangle AQN: and, if the motion of *qn* was continued uniformly from the term when *q* comes to *d*, it would deſcribe a ſpace equal to the parallelogram EG in the ſame time Q deſcribes DG, by the firſt caſe, (art. 85. & 86.) Therefore, ſince the velocity of Q is conſtant, the velocity of *qn* increaſes or decreaſes as DE, or as AD increaſes or decreaſes; and, conſequently, it is accelerated or retarded in a continued manner. It is evident, that *aq* is determined from AQ in the ſame manner as *ap* is determined from AP. Therefore, ſince the points P and Q are ſuppoſed to come to D with equal velocities, the points *p* and *q* ſhall come to *d* with equal velocities, by theor. 11. and the velocity of *pm* at that term is equal to the velocity of *qn*. From which it follows, that if the motion of *pm* was continued uniformly from the term when *p* comes to *d*, it would deſcribe a ſpace equal to the parallelogram EG, in the ſame time that the point P would deſcribe DG with its motion continued uniformly from the ſame term. Therefore, in general, the motion with which the baſe AD flows would generate DG, and the motion with which the triangle ADE flows, continued uniformly,

uniformly, would generate a ſpace equal to the parallelogram EG, in the ſame time: ſo that the fluxion of the baſe AD is repreſented by DG, and at the ſame time the fluxion of the triangle ADE by the parallelogram EG.

91. Corollary I. The motion with which the triangle flows is the ſame, whether it increaſe or decreaſe with an uniform or with a variable motion, when the baſe is of the ſame magnitude, and flows with the ſame motion. For, when AD and DG are given, the parallelogram EG is of a given magnitude.

92. Cor. II. The triangle ADE, any trapezium ADEF when the line FE is given in poſition, and the parallelogram ADEL when the ſide AL is invariable, flow with the ſame motion when the motion with which their baſe AD flows is given. For all thoſe motions continued uniformly would generate the ſame parallelogram EG in the ſame time. Fig. 21.

93. Cor. III. While the baſe AD by increaſing uniformly acquires the augment DG, the triangle ADE acquires the augment DGHE. But it is only the part EG of this augment that can be ſaid to be generated by the motion with which the triangle ADE flows at the term when P comes to D. The part EIH is generated in conſequence of the accelerations of this motion: for the ſpace EG is all that would be generated by the motion with which the triangle ADE flows, if that motion was continued uniformly, without any further acceleration, for the time in which the baſe AD acquires the augment DG. If the motion of *pm* was to be accelerated no more after it arrives at *d*, then we have ſhewn that *pm* proceeding with an uniform motion would deſcribe a ſpace equal to EG, and not to the trapezium DEHG, in the time P deſcribes DG; and, in meaſuring this motion of *pm*, or the fluxion of the triangle ADE, the part EIH of the increment which the triangle acquires in this time ought to be rejected. When the baſe decreaſes uniformly, the triangle decreaſes with a retarded motion, the parallelogram EB, or EG, is equal to the ſpace that would be generated by this motion, or the motion of *pm* at *d*, if it was continued uniformly, while P deſcribes DB: but a leſs ſpace DECB is generated in this time by the retarded motion with which the triangle flows; and the difference ECS ariſes from

the retardation of that motion. In general, whether the motion of P be uniform or variable, whatever is generated by the motion with which the triangle flows, more than the parallelogram EG, (in the time P would describe DG by its motion at D,) arises from the acceleration of this motion; and whatever is generated less than EG is owing to its retardation. Those accelerations or retardations may observe various laws. They depend upon the motion of the point P before or after it comes to the term D; and are different when the point E describes different right lines. But the motion with which the triangle ADE flows when P comes to D, is not affected by them, and depends upon the motion of P at the term when it comes to D and the magnitude of the right line DE only, the angle EDG being given.

FIG. 18. 94. COR. IV. When the base increases uniformly, the triangle increases with a motion that is uniformly accelerated. For, if DG be described by P with any uniform motion in a given time, DG will be of an invariable magnitude, and the parallelogram EG will increase in the same proportion as DE or AD, and therefore will increase uniformly; so that the velocity of *pm*, or of the point *p*, will be as the time from the beginning of the motion, supposing the point P to begin its motion at A. When the base decreases uniformly, the motion with which the triangle flows is uniformly retarded: for it decreases in the same proportion as DE or AD decreases; and the velocity of *pm*, or of the point *p*, decreases in the same proportion as the time that remains to flow till P come to A. The motion of *pm*, or of *p*, in the first case, is similar to that of heavy bodies descending by the action or influence of an uniform gravity; and the motion of *pm*, if it was to be accelerated no more after it comes to *d*, would be similar to the motion of a heavy body, if, after a like term, the action of gravity upon it was to cease, or was destroyed by an equal opposite action, pressure or resistance. And thus it appears, that we have made no suppositions, in demonstrating this proposition, but such as are not only easily conceived (which however is all that is required in geometry) and generally admitted, but are also founded in nature and common observation, and are illustrated by the motions that are most universally known. When the base AD decreases

decreaſes uniformly, the motion of *pm*, or of the point *p*, is ſimilar to the motion of a heavy body riſing in a line perpendicular to the horizon againſt the action of an uniform gravity.

95. Cor. V. It may be worth while to obſerve, that the doctrine of motions that are accelerated or retarded uniformly, is eaſily demonſtrated from the two firſt caſes of this propoſition, without ſuppoſing any quantities indiviſible or infinitely diminiſhed. When the motion of P is uniform, AP repreſents the time; the ſpace deſcribed by *pm*, being always equal to the triangle APM, increaſes in the duplicate ratio of the time AP, (Elem. 19. 6.) and the ſpace deſcribed by the point *p* increaſes in the ſame proportion. The velocity of *p* increaſes uniformly, in the ſame proportion as AP the time from the beginning of the motion, by the laſt corollary. From which it follows, con- Fig. 21.
verſely, that if the motion of any point, as p, be accelerated uniformly, or in the ſame proportion as AP the time from the beginning of the motion, the ſpace deſcribed by it from the beginning of the motion ſhall increaſe in the duplicate ratio of this time. For, if the velocity of p be equal to the velocity of *p* at any term of the time, their velocities, and conſequently the ſpaces deſcribed by them in the ſame time, (by theor. 4.) muſt be always equal. And this is the celebrated theorem diſcovered by Galileus. Becauſe the parallelogram EG is to the parallelogram AE as the baſe DG is to AD; it appears, that, in the time P deſcribes AD, the right line *pm* would deſcribe a ſpace equal to the parallelogram AE, if the motion it acquires at *d* was continued uniformly. But the parallelogram AE is double of the triangle ADE, or of the parallelogram *ae*. Therefore the point *p* would deſcribe a line double of *ad*, if the motion it acquires at *d* was continued uniformly, in the ſame time that it deſcribes *ad* with a motion that is accelerated uniformly: which is another of his theorems. Becauſe the trapezium BCHG is double of the parallelogram EG, it follows, that the ſpace deſcribed by a motion that is uniformly accelerated, is equal to the ſpace that would be deſcribed in an equal time by the motion at the middle term of that time continued uniformly; as was ſhewn in a different manner in art. 72. When the baſe AP decreaſes uniformly, the velocity of *p* from *o* towards *a* decreaſes

uniformly, and may be always represented by AP. The space *pa* that remains to be described by the point *p* before its motion be at an end, decreases in the same ratio as the triangle APM decreases, or in the duplicate ratio of AP; that is, in the duplicate ratio of the velocity of *p*, or of the time that remains to flow till the end of the motion. Therefore the spaces that may be described by bodies before their motions be destroyed, when they are continually retarded by an uniform gravity, (or by any uniform power or resistance that diminishes their velocities equally in any equal times,) are in the duplicate ratio of their velocities.

96. Cor. VI. If the angle APM be a right one, and PM be double of AP, the triangle APM shall be always equal to a square described upon AP; and the motions with which this square and the triangle APM flow shall be always equal, by theor. 3. Therefore, when P comes to D, the motion with which a square upon AD flows, continued uniformly, would generate a rectangle equal to EG, in the time P describes DG. But the rectangle EG is equal, in this case, to a rectangle contained by 2AD and DG; and this rectangle represents the fluxion of the square, when DG represents the fluxion of AD the side of the square. The parallelogram *ad* being equal to the square of AD, the right lines *af*, AD and *ad* are in continued proportion; and, the rectangles *ek* and EG being equal, DG (which represents the fluxion of AD) is to *dk* (which represents the fluxion of *ad*) as *af* is to DE, or 2AD; and DG is to one half of *dk* in the subduplicate ratio of *af* to *ad*. Therefore, when three quantities are in continued proportion, and the first is invariable, the fluxion of the second term is to the fluxion of the third, as the first term is to twice the second term; and the fluxion of the second term is to one half of the fluxion of the third term in a subduplicate ratio of the first term to the third.

97. Cor. VII. When the base flows uniformly, the velocity of the motion with which the triangle flows increases in the same proportion as PM or AP increases, and is equally augmented in any equal times. Therefore the fluxion of this velocity, or the second fluxion of the triangle, is constant: and if we conceive the continual acceleration of this velocity to be the

effect

effect of ſome power, the exertion of this power muſt be ſuppoſed uniform. When P comes to D, the motion with which the triangle flows, continued uniformly, would generate the parallelogram EG, in the ſame time P deſcribes DG. When P comes to G, the motion with which the triangle then flows, continued uniformly, would generate the parallelogram DH in the ſame time. The difference of theſe parallelograms, or the parallelogram IN, repreſents the ſecond fluxion of the triangle ADE; and IN is double of the triangle EIH, which is the part of the increment of the triangle that is generated in conſequence of the uniform acceleration of the motion with which it flows while P deſcribes DG. It is evident, that, if the motion of P be increaſed or diminiſhed, then the motion with which the triangle ADE flows, or its fluxion, is increaſed in the ſame proportion; becauſe the parallelogram EG increaſes in the ſame proportion as its baſe DG. But the triangle EIH, or the parallelogram IN which meaſures the ſecond fluxion of the triangle, increaſes or decreaſes in the duplicate ratio of DG, or of the ſpace EG which meaſures its firſt fluxion. And, the velocity of P remaining, if the time in which DG is ſuppoſed to be deſcribed be continually diminiſhed, then EG which meaſures the firſt fluxion of the triangle decreaſes in the ſame proportion; but IN is diminiſhed in the duplicate ratio of this time, and its ratio to EG may become leſs than any aſſignable ratio. The parallelogram IN (which meaſures the ſecond fluxion of the triangle ADE) is equal to the difference of the increments DECB, DGHE which are generated while P deſcribes BD and DG. The increment DGHE is equal to the parallelogram EG (which meaſures the firſt fluxion of the triangle) added to one half of IN which meaſures its ſecond fluxion. Theſe things agree with the 74th, 75th, 76th and 77th articles, and illuſtrate them. When the baſe flows uniformly, it is evident that the triangle has no third fluxion: and what we have ſaid of the triangle APM is eaſily applied to the right line *ap*, which always flows in the ſame manner, becauſe the rectangle contained by it and the invariable right line *af* is always equal to the triangle, by the ſuppoſition.

LEMMA

LEMMA IV.

98. *Let the points* P *and* Q *deſcribe the right lines* AO, AV *given in poſition, with motions that are either uniform, or are always to each other in an invariable ratio; and, the parallelogram* APRQ *being completed, the point* R *ſhall be always found in the ſame right line.*

FIG. 22. & 23.

While the point P deſcribes BD and DG upon the line AO, let Q deſcribe KL and LM upon the line AV; complete the parallelograms ABCK, ADEL and AGHM; let CN parallel to AO meet DE and GH in S and N; and let EI parallel to AO meet GH in I. Becauſe the velocities of P and Q are to each other in an invariable ratio, it follows from theor. 6. that BD is to DG as KL is to LM. Therefore BD is to BG as KL is to KM, or CS is to SE as CN is to NH; and, conſequently, any three places C, E and H of the point R being in a right line, it follows, that the point R is always found in the ſame right line.

PROP. III.

99. *The fluxions of the right lines* AD *and* AL *being repreſented by* DG *and* LM, *the fluxion of the rectangle* AE, *contained by* AD *and* AL, *is meaſured accurately by the ſum of the rectangles* EG *and* EM *when theſe lines increaſe or decreaſe together, but by the difference of* EG *and* EM *when one of thoſe lines decreaſes while the other increaſes.*

FIG. 22. & 23.

FIG. 22. *Caſe* 1. Let the ſides AP, AQ of the rectangle AR both increaſe, or both decreaſe, together, by the uniform motions of the points P and Q. When P comes to D, let Q come to L, and R to E; and, while P deſcribes DG, let Q deſcribe LM, and R deſcribe EH: and the point R ſhall be always found in the right

right line EH, by the laſt lemma. Produce HE, and let it meet AD in F before it meets with AL: then, the rectangle AR being always equal to the ſum of the triangle FPR and trapezium AFRQ, the increment or decrement of the rectangle is always the ſum of the increments or decrements of the triangle and trapezium that are generated in the ſame time; and, by theor. 7. the motion with which the rectangle flows, is always equal to the ſum of the motions with which that triangle and trapezium flow. But, by prop. 2. (art. 85.) if the motions with which the triangle FDE and trapezium AFEL flow were continued uniformly, the parallelograms EG and EM would be generated by them in the ſame time that the points P and Q deſcribe DG and LM with their uniform motions. Therefore, if the motion with which the rectangle AE flows was continued uniformly, a ſpace equal to the ſum of the parallelograms EG and EM would be generated by it in the ſame time; and, if the fluxions of the ſides AD and AL be repreſented by DG and LM, the fluxion of the rectangle AE muſt be repreſented by the ſum of the rectangles EG and EM. The demonſtration is the ſame when HE produced meets AL before it meets with AD: only, in this caſe, EM repreſents the fluxion of the triangle, and EG the fluxion of the trapezium; and, if HE produced paſs through A, the point F coincides with A, and the rectangle AE is the ſum of two triangles the fluxions of which are repreſented by the rectangles EG and EM.

100. *Caſe* 2. Let the ſide AQ decreaſe while the ſide AP increaſes; and, the motions of P and Q being ſtill uniform, the point R ſhall deſcribe a right line EH, by the laſt lemma. Becauſe the angle MHE is acute, and the angle HML right, HE produced ſhall meet with AL produced beyond L in ſome point *f*; and it ſhall meet with AD produced beyond D in ſome point F. The rectangle AE in this caſe is always the exceſs of the trapezium ADE*f* above the triangle *f*LE; and the motion with which it flows is always equal to the difference of the motions with which that trapezium and triangle flow, by art. 41. Therefore, if the fluxion of the ſides AD and AL be repreſented by DG and LM, the fluxion of the rectangle AE ſhall be repreſented by the difference of the rectangles EG and EM. FIG. 23.

101. *Caſe*

101. *Case* 3. When the motions of the points P and Q are variable, but are in an invariable ratio to each other, the point R ſtill deſcribes a right line, by the laſt lemma; and the motion with which the rectangle AE flows, is ſtill meaſured by the ſum or difference of the rectangles EG and EM when the motions of P and Q, at the term when they come to D and L, are meaſured by DG and LM, by prop. 2. In general, let
FIG. 24. the motions of P and Q be varied at pleaſure; and let them come to D and L with motions by which continued uniformly they would deſcribe DG and LM in any given time. Let the right line AC conſtitute with AD an angle that is half a right one; produce ED and EL till they meet AC in C and K: and KE, or CE, ſhall be always equal to the ſum of AD and AL. Upon AD produced let AF be always equal to CE, and FR parallel to DE meet AC in R; and the triangle AFR (which is equal to KEC) ſhall always exceed the rectangle AE by the two triangles ADC, ALK. Therefore the motion with which the rectangle AE increaſes, is equal to the exceſs of the motion with which the triangle AFR increaſes above the ſum of the motions with which the triangles ADC, ALK increaſe. When the ſides AD, AL increaſe together, take FT equal to the ſum of DG and LM, but equal to their difference when AL decreaſes while AD increaſes. Let GQ and TX, parallel to DE, meet AC in the points Q and X; let MY, parallel to AD, meet AC in Y; let CP and RZ, parallel to AD, meet GQ and TX in P and Z; and let KS, parallel to AL, meet AC in Y. By prop. 2. if the motion with which the triangles AFR, ADC, ALK increaſe were continued uniformly, they would generate the ſpaces FZ, DP, LS in the ſame time that the motions with which their baſes increaſe, continued uniformly, would generate the right lines FT, DG, LM. But, when the ſides AD, AL increaſe together, FT is equal to the ſum of DG and LM; and FR is equal to KE, or EC: and therefore FZ is equal to the ſum of the rectangles ES and EP. From which ſum deduce DP and LS, which meaſure the motions with which the triangles ADC, ALK increaſe, and the remainder is the ſum of EG and EM; which therefore meaſures the motion with which the rectangle AE flows. But if AL decreaſe while AD increaſes,

ſes, then, FT being equal to the difference of DG and LM, the rectangle FZ will be equal to the difference of the rectangles EP and ES; and, becauſe the triangle ALK decreaſes, the motion with which the ſum of the triangles ADC, ALK increaſes will be meaſured by the difference betwixt DP and LS. From FZ, or the difference betwixt EP and ES, ſubduct the difference betwixt DP and LS, and the remainder will be equal to the difference betwixt EG and EM; which therefore meaſures the motion with which the rectangle AE flows when DG and LM meaſure the motions with which its ſides AD and AL flow. If DG be equal to LM, the parallelogram EP is equal to ES; the line FT, and the motion with which the triangle AFR flows, vaniſh; the motion with which the rectangle AE increaſes is equal to the motion with which the ſum of the triangles ADC, ALK decreaſes, and therefore is meaſured by the difference betwixt LS and DP; which is equal to the difference betwixt EG and EM, becauſe ES the ſum of EM and LS is equal to EP the ſum of EG and DP. If LM be greater than DG, then the triangle KEC (or AFR) decreaſes, and FT muſt be taken from F towards A: but the motion with which AE increaſes will ſtill be meaſured by the exceſs of EG above EM. When EG is equal to EM, then this motion vaniſhes; and, when EG is leſs than EM, the rectangle AE decreaſes by a motion that is ſtill meaſured by their difference. Thus it appears, that, whether the ſides AD and AL flow with uniform or variable motions, and the point E deſcribe a right line or not, when the fluxions of the ſides AD and AL are meaſured by DG and LM, the fluxion of the rectangle AE is meaſured by the ſum or difference of the rectangles EG and EM, their ſum when the ſides increaſe or decreaſe together, their difference when one ſide decreaſes while the other increaſes. We ſhall demonſtrate this propoſition in a different manner afterwards. Fig. 25.

102. Cor. When the ſides of the rectangle AE increaſe with uniform motions, the rectangle flows with a motion that is uniformly accelerated: For the rectangle is always equal to the ſum of the triangle FDE and trapezium AFEL; both of which increaſe with motions that are uniformly accelerated, by cor. 4. prop. 2. While the ſides AD and AL acquire the augments DG Fig. 22.

and LM, the rectangle AE is increaſed by the ſpace DELMHG: but there is no more than the parts EG and EM of this increment generated in conſequence of the motion with which the rectangle AE flows at the term or moment when P comes to D and Q to L. The part *Ib* is generated in conſequence of the uniform acceleration of this motion while the points P and Q deſcribe DG and LM. This part is rejected in meaſuring the motion with which the rectangle AE flows at that term; but the increaſe or acceleration of this motion generated in a given time, or the uniform power by which this increaſe may be conceived to be produced, may be meaſured by it. A ſpace double of *Ib* would be generated in the ſame given time by an uniform motion equal to the difference of the motions with which the rectangles AH and AE flow; and therefore the ſecond fluxion of the rectangle AE may be meaſured by that ſpace; as was demonſtrated in the fifteenth theorem.

103. In the ſecond caſe, the fluxion of the rectangle AE may be expreſſed by the rectangle that is contained by LM and the difference of FD and AD, or of AF and 2AD; for that rectangle is equal to the difference of the rectangles EG and EM. In this caſe, the fluxion of the rectangle AE continually decreaſes while AD increaſes; it vaniſhes when AD is equal to DF or to one half of AF; and the rectangle AE is greateſt at that term. When AD becomes greater than FD, the fluxion of the rectangle AE becomes negative; that is, the rectangle decreaſes continually till it vaniſh when AD becomes equal to AF. The ſecond fluxion of the rectangle is invariable, and is meaſured by 2*Ib*, as in the firſt caſe: but it is conſidered as negative, becauſe it continually diminiſhes the firſt fluxion, or retards the motion with which the rectangle increaſes till AD become equal to one half of AF, and accelerates the motion with which the rectangle decreaſes after that term. If we ſuppoſe, Fig. 22. in the firſt caſe, the points P and Q to move from O and V towards A, and their motions to be continued after P comes to F and Q to A, we ſhall have the firſt and ſecond caſes comprehended in one view.

104. We have inſiſted ſo much on the two preceeding propoſitions for the reaſons mentioned in the 69th and 85th articles.

ticles. In delivering thoſe elements, we have endeavoured to avoid every thing that might appear abſtruſe or obſcure, and to obviate the objections that have been advanced lately againſt this doctrine, though we have not taken particular notice of them. In the third lemma, we demonſtrated that the motion with which the triangle flows is perpetually accelerated while the baſe flows uniformly. Nor do we ſee how this can be diſputed, ſince that motion cannot be ſuppoſed to be either uniform or retarded for any part of the time how ſmall ſoever. Some Philoſophers may be of opinion, that a line or velocity cannot be conceived to increaſe or decreaſe continually, but by certain ſmall indiviſible increments or decrements only. How repugnant ſoever this principle may be to what is demonſtrated by Geometricians, the preceeding propoſitions and a great part of this doctrine may be eaſily adapted to it, as we hinted in the 7th article. It was in effect in this form that ſome part of its elements firſt appeared in the method of indiviſibles: but we proceed to eſtabliſh it on the more accurate principles of the ancients.

CHAP. III.

Of the Fluxions of plane curvilineal Figures.

LEMMA V.

105. *THE baſe of any figure being ſuppoſed to increaſe uniformly, the area flows with an accelerated or retarded motion, according as the ordinates increaſe or decreaſe while the baſe increaſes.*

While the point P deſcribes the baſe by moving from A towards *a*, or from *a* towards A; let the right line PM, by moving parallel to itſelf from AF towards *af*, or from *af* towards AF, generate the area APMF or *a*PM*f*. Let *q*D and DQ be equal FIG. 26.

equal parts of the base described by P in any equal times; let *qn*, DE and QN, parallel to PM, meet the curve, or any continued line EE*f*, in *n*, E and N: Then it is manifest, that, when the ordinates from *qn* to QN increase, the area DENQ is greater than the area D*Enq*. Therefore, when AP increases uniformly, or the point P describes *q*D before it describes QD, and the right line PM generates the area *q*DE*n* before it generates DQNE, the motion with which the area flows is always accelerated: but if *a*P increase uniformly, or the point P move from Q to *q*, the increments of the area *a*PM*f* which are generated in any equal times that succeed after one another, perpetually decrease, and the motion with which it flows is always retarded.

106. It is evident, in the same manner, that, when the ordinates decrease while the base AP decreases, the area APMF decreases with a retarded motion: but, when the ordinates increase while the base *a*P decreases, the area *a*PM*f* decreases with
FIG. 27. an accelerated motion. It is also manifest, that, when the area APMF is supposed to flow with an uniform motion, so that any increments or decrements *qn*ED, DENQ generated in equal times are always equal, then *q*D is greater than DQ; and the base AP increases with a retarded motion, or decreases with an accelerated motion: but the base *a*P increases with an accelerated motion, or decreases with a retarded motion. It is easy to represent the motion with which the area flows by the motion of a given right line *pm*, that, by moving parallel to itself, may be supposed to generate a parallelogram always equal to the area, as in the preceeding chapter. We omit this right line *pm* in the following propositions; but it may be easily supplied by the reader if he pleases.

PROP. IV.

107. *The fluxion of the base* AD *being represented by* DG, *the fluxion of the area* ADEF *is accurately measured by the parallelogram* EG.

FIG. 26. First, let the base AP increase uniformly, and the ordinates PM increase while the base increases. The area flows, in this case,

caſe, with an accelerated motion, by the laſt lemma; and, if this motion was continued uniformly from the term or moment when P comes to D, a ſpace would be generated by it greater than BDEC, (by ax. 2.) but leſs than DGHE, (by ax. 1.) in the ſame time P deſcribes DG with its uniform motion. Let this ſpace be ſuppoſed equal to X, and X ſhall be equal to the parallelogram EG. For, if it be ſaid to be greater than EG, let DR be greater than DE in the ſame ratio; complete the parallelogram DRLG, and it ſhall be equal to X, becauſe it is to EG as DR is to DE, (Elem. 1. 6.) or as X is to EG. Let RL meet the curve CH in N, and NQ parallel to DE meet the baſe in Q. Then, ſince the parallelogram RG is to RQ as the baſe DG is to DQ, it follows from the firſt general theorem, (art. 16.) that a ſpace equal to the parallelogram RQ would be generated by the motion with which the area APMF flows, continued uniformly from the term or moment when P comes to D, in the ſame time P deſcribes DQ. But, while P deſcribes the line DQ, the area DENQ is generated by the accelerated motion with which the area flows; and DENQ muſt be greater than the ſpace which would have been generated in this time if the motion with which the area flows had been continued uniformly from the beginning of the time without any acceleration, by the firſt axiom. Therefore the area DENQ is greater than the parallelogram RQ. But, becauſe DR is greater than DE, (in the ſame proportion as X is ſuppoſed to be greater than EG,) and the ordinates from DE to QN continually increaſe, the parallelogram RQ is greater than DENQ. And theſe being contradictory, it follows that the ſpace X is not greater than EG. If X be ſaid to be leſs than EG, let D*r* be leſs than DE in the ſame proportion; complete the parallelogram D*r*IG, and it ſhall be equal to X. Let *rl* produced meet the curve in *n*, and *nq* parallel to DE meet the baſe in *q*. Then, ſince the parallelogram *r*G is to *rq* as the baſe DG is to D*q*, it follows, that a ſpace equal to the parallelogram *rq* would be generated by the motion with which the area flows, continued uniformly from the term when P comes to D, in the ſame time P deſcribes a line equal to D*q* with its uniform motion. But, while P deſcribed *q*D before it came to D, the ſpace *qn*ED was generated by

by the accelerated motion with which the area flows; and this ſpace *qn*ED muſt be leſs than the ſpace which would be generated by the motion with which the area flows if it was continued uniformly from the term when P comes to D, by the ſecond axiom; that is, *qn*ED is leſs than the parallelogram *nq*: which is abſurd. Therefore the ſpace X is neither greater nor leſs than the parallelogram EG, but preciſely equal to it. When the baſe decreaſes or increaſes uniformly, and the ordinates decreaſe at the ſame time, the area flows with a retarded motion; and the demonſtration is the ſame as that of the 87th article.

FIG. 27. 108. Let the area APMF flow uniformly, and acquire the augment DESK in any given time. Let DG be the ſpace that would be deſcribed by P with its motion at D continued uniformly for the ſame time; and the area DESK ſhall be equal to the parallelogram EG. For, if the ordinates increaſe while the baſe increaſes, the motion of P is retarded, (by art. 106.) and, if the area DESK be ſuppoſed equal to any parallelogram DRLG greater than EG, let RL meet the curve EH in N, and NQ parallel to ED meet the baſe in Q. Then, becauſe the parallelogram RG is to RQ as the baſe DG is to DQ, it follows, that the point P, with its motion continued uniformly from D, would deſcribe the line DQ, in the ſame time that the area by flowing uniformly acquires an augment equal to the parallelogram RQ; and the point P would deſcribe a leſs line than DQ, by the ſame uniform motion, while the area by flowing uniformly acquires the augment DENQ. But, becauſe the motion of P is retarded while the area increaſes uniformly, it follows from the third axiom, that the point P would deſcribe a greater line than DQ, by its motion continued uniformly from D, while the area acquires the augment DENQ. And theſe being contradictory, the area DESK is not greater than EG. In like manner, it appears from the fourth axiom, that the area DESK is not equal to any parallelogram D*r*/G leſs than EG. Therefore the area DESK is equal to the parallelogram EG. When the ordinates decreaſe while the baſe either increaſes or decreaſes, and the area is ſuppoſed to flow uniformly, the baſe flows with an accelerated motion; and the demonſtration is the ſame as in the 89th article. In thoſe caſes, when the motion with

which

which the area flows is uniform, the parallelogram EG (which is equal to the ſpace that would be generated by that uniform motion in a given time) is of an invariable magnitude; and the velocity of the point P is always reciprocally as the ordinate PM: whence it may be determined, whether the motion with which the area flows is accelerated or retarded, when the law according to which the velocity of P varies and the nature of the curve FEH are known.

109. In general, whatever the motion of the point P may be, or that with which the baſe flows, the propoſition is demonſtrated univerſally from art. 107. by the ninth and eleventh general theorems in the manner that was deſcribed in the 90th article. Therefore, when the fluxion of the baſe AD is repreſented by DG, the fluxion of the area ADEF is repreſented by the parallelogram EG. And, converſely, when the fluxion of the area is repreſented by the parallelogram EG, the fluxion of the baſe is repreſented by DG.

110. Cor. I. When the fluxion of the baſe AD is given, the fluxion of the area ADEF is the ſame, whatever line be deſcribed by the point M if it paſs through E. If the baſe flow uniformly, whatever is generated more or leſs than the parallelogram EG by the motion with which the area flows, in the time P deſcribes DG, proceeds from the acceleration or retardation of the motion with which the area flows at the term when P comes to D.

111. Cor. II. When the fluxions of the baſes of two figures that are generated in the ſame time are equal, the fluxions of the areas are as their ordinates; and when the ordinates are always in a given ratio, (as in the caſe deſcribed in the introduction, pag. 7. fig. 1.) the areas that are generated in the ſame time are in the ſame given ratio, by the ſixth general theorem. When the fluxions of the baſes are reciprocally as the ordinates, the fluxions of the areas are equal; and when the fluxions of the baſes are always to each other in this ratio, the areas that are generated in the ſame time are equal, by the fourth general theorem.

112. Cor. III. Let AE be a rectangle contained by the two ſides AD, AL; while AD and AL flow with any accelerated Fig. 28.

or

or retarded motions, let E describe any line FE that meets one of the sides AD in F. Then, the rectangle AE being equal to the sum or difference of the areas AFEL and FDE, the fluxion of the rectangle AE shall be measured by the sum or difference of the rectangles EG and EM, when the fluxions of the sides AD, AL are measured by the right lines DG and LM. And thus the third case of the third proposition is briefly demonstrated.

Fig. 29. 113. Cor. IV. If the area AD*ef* be always equal to the rectangle contained by the ordinate DE and the right line DG that represents the fluxion of the base AD, then shall the parallelogram *e*G measure the fluxion of the area AD*ef*, or of the parallelogram EG, or the second fluxion of the area ADEF. And, if the area ADef be always equal to the parallelogram *e*G, the parallelogram eG shall measure the third fluxion of the area ADEF. In the same manner the higher fluxions of this area may be represented; but, in some cases, by proceeding thus, we shall come to some fluxion that is constant, which cannot be considered as a flowing or variable quantity.

114. Cor. V. From this proposition some general theorems in the doctrine of motion may be easily demonstrated. If the
Fig. 26. & 30. base AP represent the time of any motion, PM the velocity at any term of the time P, the space described by the motion in the time AP may be measured by the area APMF; that is, the space described in the time AP shall be to the space described in any other time AD as the area APMF is to the area ADEF. For, the motion of P being uniform, the motion with which the area APMF flows varies in the same proportion as the ordinate PM, by what has been demonstrated; and therefore, if the invariable right line *pm*, by moving parallel to itself, describe a space *am* always equal to the area APMF, the velocity of the right line *pm*, or of the point *p*, shall be always measured by the ordinate PM at any term of the time, as P. From which it follows, conversely, (as in art. 95.) that if we suppose the velocity of the point *p* at any term P to be measured by the ordinate PM, the space described by the point *p* in the time AP shall be measured by the area APMF; or, the space described by *p* in the time AP shall be to the space described by it in the time AD as the area APMF is to the area ADEF. In

Pl. IV. Pa. 136.

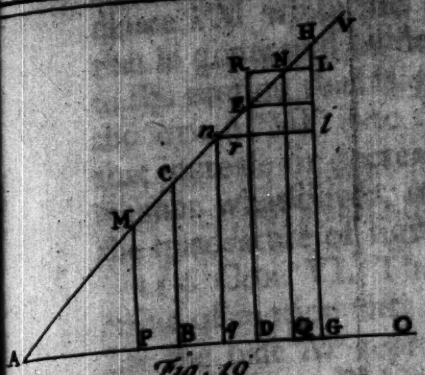
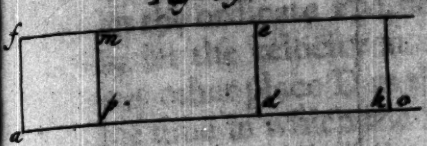


Fig. 19



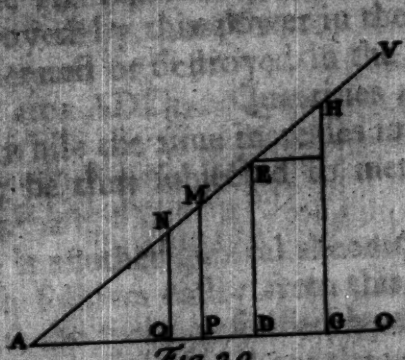
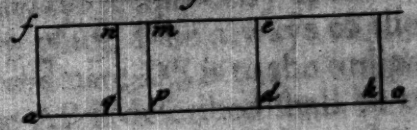


Fig. 20



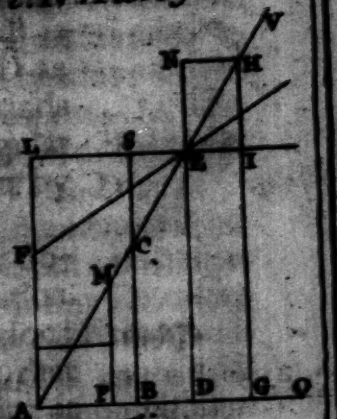
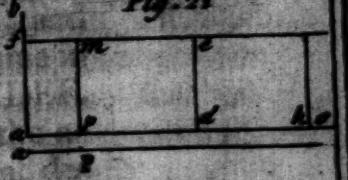


Fig. 21



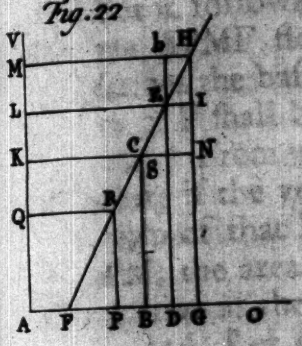


Fig. 22

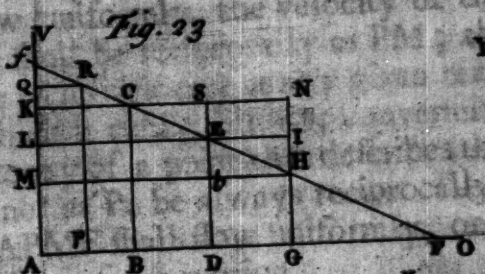


Fig. 23

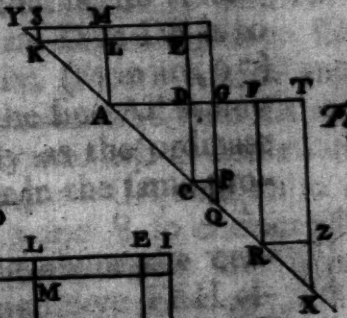


Fig. 24

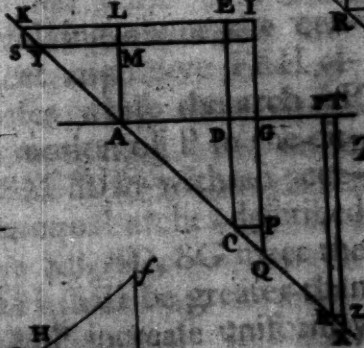


Fig. 25

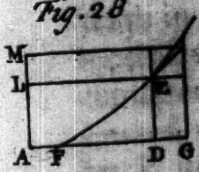


Fig. 28

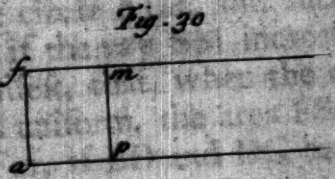


Fig. 30

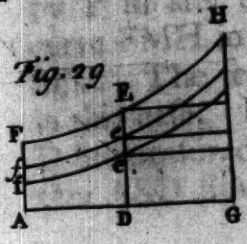


Fig. 29

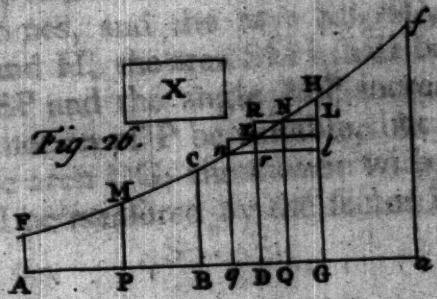


Fig. 26.

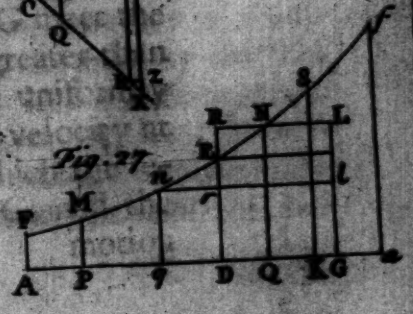


Fig. 27

the ſame manner, if the action or exertion of any power that accelerates or retards motion be always repreſented by the ordinate PM while the baſe AP repreſents the time, the motion that is generated or deſtroyed by this power in the time AP is to the motion that is generated or deſtroyed in the time AD as the area APMF is to the area ADEF. Quantities of all kinds that increaſe or decreaſe while the time increaſes in any regular manner whatſoever, may be thus ſubjected to menſuration by the quadrature of figures.

115. Cor. VI. There is another general theorem in the doctrine of motion that alſo follows eaſily from this propoſition. Let the baſe AP repreſent the ſpace deſcribed by any motion; let the ordinate PM be always reciprocally as the velocity; that is, let the velocity at any place P be always to the velocity at any other place D as the ordinate DE is to the ordinate PM; and the time in which the line AP is deſcribed ſhall be to the time in which AD is deſcribed as the area APMF is to the area ADEF. For it follows from this propoſition, (art. 108.) that, if the area APMF flow uniformly, the velocity of the point that deſcribes the baſe ſhall be reciprocally as PM; that is, its velocity at P ſhall be to its velocity at any given term D as DE is to PM. From which it eaſily follows, converſely, (as in art. 95.) that, if the velocity of a point that deſcribes the line AP at any term of that line, as P, be always reciprocally as the ordinate PM, the area APMF ſhall flow uniformly, or in the ſame proportion as the time of the motion. Fig. 27.

116. Let a given right line SP, revolving about the center S, generate the circle AD*a*, and meet any curve FEH always in M; then, if the ray SM increaſes while the arch AP increaſes, it is manifeſt, that, when the motion of P in the circumference AD*a* is uniform, the area FSM flows with an accelerated motion. For, if BD and DG be equal arches deſcribed by P in any equal times, and the rays SB, SD, SG meet the curve FMH in C, E and H, the area SEH ſhall be greater than SEC. The ſector ASP and the angle ASP increaſe uniformly in this caſe. If the motion of P be variable, but its velocity at D be meaſured by the arch DG, the motion with which the ſector ASD flows ſhall be meaſured by the ſector DSG, and the Fig 31

motion with which the angle ASD flows by the angle DSG; as may be demonſtrated in the ſame manner as the 80th article.

PROP. V.

117. *The area* FSE *being ſuppoſed to flow as in the laſt article, let the arch* EI *deſcribed from the center* S *meet the ray* SG *in* I; *and, if the fluxion of the ſector* ASD *be repreſented by the ſector* DSG, *the fluxion of the curvilineal area* FSE *ſhall be accurately meaſured by the ſector* ESI.

FIG. 31.

Let the motion of P in the circumference AD*a* firſt be uniform, and the rays SM terminated by the curve FMH increaſe while AP increaſes, as in the laſt article; and the area FSM ſhall flow with an accelerated motion. Suppoſe that a ſpace equal to X would be generated by this motion continued uniformly from the term when P comes to D, in the ſame time P deſcribes DG; and the ſpace X ſhall be leſs than the area ESH, (by ax. 1.) but greater than the area CSE, (by ax. 2.) If X be not equal to the ſector SEI, let it firſt be greater than that ſector. Produce SE to R till SR be to SE in the ſubduplicate ratio of the ſpace X to SEI; let the arch RL deſcribed from the center S meet the rays SD, SG in R and L, and the curve in N; and the ſpace X ſhall be equal to the ſector SRL. Let SN meet the circle in Q; and, ſince the ſector SRL is to SRN as DG is to DQ, it follows, (by theor. 1.) that, if the motion with which the area FSE flows be continued uniformly, it will generate a ſpace equal to the ſector SRN, in the time that P deſcribes the arch DQ. But this ſpace muſt be leſs than the area SEN which is generated in the ſame time by the accelerated motion with which the area FSM flows, by the firſt axiom. Therefore the ſector SRN is leſs than the area SEN. But this is abſurd; and, conſequently, the ſpace X is not greater than the ſector SEI. If it be ſaid that the ſpace X is leſs than the ſector SEI, let S*r* be leſs than SE in the ſubduplicate ratio of the ſpace X to the ſector SEI. From S as center deſcribe the arch *rl*, meet-

meeting the rays SD, SG in *r* and *l*, and the curve EC in *n*. Let S*n* meet the circumference of the circle AD*a* in *q*; and the ſpace X ſhall be equal to the ſector S*rl*. But S*rl* is to S*rn* as *rl* is to *rn*, or DG to D*q*; and therefore (by theor. 1.) a ſpace equal to the ſector S*rn* would be generated by the motion with which the area FSE flows continued uniformly, during the time in which P deſcribes an arch equal to *q*D; and this ſpace muſt be greater than the area S*n*E which is generated by the accelerated motion with which the area flows while P deſcribes *q*D before it comes to D, by the ſecond axiom; that is, the ſector S*nr* muſt be greater than the area S*n*E: which is abſurd. Therefore the ſpace X is neither greater nor leſs than the ſector SEI, but is preciſely equal to it.

118. The reſt remaining as in the laſt article, ſuppoſe the area FSM to increaſe uniformly, and to acquire the equal increments CSE, ESH in any equal times that immediately ſucceed after each other. Let the rays SC and SH meet the circumference AD*a* in *k* and K; and, ſince the rays from S terminated by the curve CEH are ſuppoſed to increaſe continually from SC to SH, it is manifeſt, that the angle ESH is leſs than ESC, and that the arch DK is leſs than D*k*. Therefore the motion of the point P, in deſcribing the circumference AD*a*, is perpetually retarded. Suppoſe that the point P would deſcribe the arch DG by its motion at D continued uniformly, in the ſame time that the area FSE by flowing uniformly acquires the augment ESH; and the area ESH ſhall be equal to the ſector ESI. For, if ESH be greater than ESI, let SR be greater than SE in the ſubduplicate ratio of ESH to ESI; let the arch RL deſcribed from the center S meet SG in L, and the curve EH in N; and let SN meet the circumference DG in Q. Then, ſince the ſector SRL is to the ſector SRN as the arch RL is to the arch RN, or as DG is to DQ, it follows, that, in the time the area FSE by flowing uniformly acquires an augment equal to the ſector SRN, the point P would deſcribe DQ with its motion continued uniformly from the term when it comes to D; and, in the time that the area FSE acquires the augment ESN, (which is leſs than the ſector SRN,) the point P would deſcribe an arch leſs than DQ by the ſame motion continued uniformly. But, FIG. 32.

while the area FSE acquires the augment ESN, the point P with a retarded motion deſcribes DQ; and it would deſcribe an arch greater than DQ in the ſame time by its motion continued uniformly from D, by the third axiom. And theſe being contradictory, it follows, that the area ESH is not greater than the ſector ESI. In the ſame manner it appears, that ESH is not equal to any ſector SIl leſs than SEI.

119. In general, all the other caſes of this propoſition are deduced from the firſt, by the ninth and eleventh theorems, in the ſame manner that we demonſtrated the 90th article: ſo that, when the fluxion of the arch AD is repreſented by DG, or the fluxion of the ſector ASD by the ſector DSG, the fluxion of the area FSE is expreſſed accurately by the ſector ESI; and, converſely, when the fluxion of the area FSE is expreſſed by the ſector ESI, produce SI till it meet the circumference ADσ in G, and DG ſhall repreſent the fluxion of the arch AD.

120. Cor. I. The fluxion of the ſector ASD is to the fluxion of the area FSE in the duplicate ratio of SD to SE. When the arch AD, or the angle ASD, flows uniformly, the fluxion of the area FSE increaſes or decreaſes in the duplicate ratio of the ray SE. But, when the area FSE flows uniformly, the fluxion of the arch AD, or of the angle ASD, is reciprocally as the ſquare of the ray SE.

Fig. 32. 121. Cor. II. In the 118th article, where the area was ſuppoſed to flow uniformly, it was ſhewn, that, if the ſector ESI be equal to the area ESH, and SI produced to the circumference ADσ meet it in G, then DG is the arch that would be deſcribed by P with its motion at D continued uniformly while the area FSE acquires the augment ESH. From this it follows, that, if the angular motion of the ray SE was continued uniformly, the angle ESI would be generated by it in the ſame time. Let SH meet the arch EI in R; and the angle ESR, which is actually generated by the ray SP revolving about the center S while the area is increaſed by ESH, is to the angle ESI that would have been generated in the ſame time if the angular motion of SE about S had been continued uniformly, as the ſector ESR is to the area ESH; and the difference of thoſe angles is to the latter angle ESI as the area ERH is to the area ESH.

122. The

122. The ſimilarity of curvilineal figures may be derived from that of rectilineal figures that are always ſimilarly deſcribed in them; or, we may comprehend all ſorts of ſimilar figures, planes or ſolids, in this general definition: Figures are ſimilar when they may be ſuppoſed to be placed in ſuch a manner, that, any right line being drawn from a determined point to the terms that bound them, the parts of the right line intercepted betwixt that point and thoſe terms are always in one conſtant ratio to each other. Thus the figures ASD, *a*S*d* are ſimilar, FIG. 33. when, any line SP being drawn always from the ſame point S meeting AD in P, and *ad* in *p*, the ratio of SP to S*p* is invariable. It is manifeſt, that the rectilineal inſcribed figures APDS, *apd*S are ſimilar, in this caſe, according to the definition of ſuch figures that is given in the Elements. If the right lines PM and *pm* parallel to each other meet SA in M and *m*; then PM ſhall be to *pm* as SM is to S*m*, or as SA is to S*a*; and AM is to *am* as PM is to *pm*. And, converſely, if AM be always to *am* as AS is to *a*S; and PM be always to *pm* as AM is to *am*, it is manifeſt, that the points S, *p* and P are always in one right line, and that S*p* is to SP in the invariable ratio of S*a* to SA. From which it appears, that the common definition of ſimilar figures, particularly that of APOLLONIUS for the conic ſections, and of CAVALERIUS for any figures whatſoever, is included in this, or is eaſily deduced from it. When the ſimilar figures are in the ſituation we have deſcribed, they are alſo ſimilarly ſituated, and all their homologous lines are either placed upon one another or parallel.

123. If the right line SP, by revolving about the given point S, generate the ſimilar figures ASP, *a*S*p*, their fluxions ſhall be to each other in the duplicate ratio of SP to S*p*, or of SA to S*a*; and, conſequently, in an invariable ratio. Therefore, by the ſixth general theorem, the ſimilar figures ASP, *a*S*p* which are generated in the ſame time are in the ſame invariable ratio. The triangles ASP, *a*S*p* and the ſegments ANP, *anp* are in the ſame duplicate ratio of SA to S*a*, or of AP to *ap*. If we ſuppoſe the ray S*a* and the figure *a*S*d* to increaſe or decreaſe ſo as to remain always ſimilar to a given figure ASD, the fluxion of the area *a*S*d* ſhall be to the fluxion of the inſcribed triangle

*a*S*d*

aSd in the invariable ratio of that area to the inſcribed triangle; and the fluxion of the area aSd increaſes or decreaſes uniformly when the fluxion of Sa is conſtant.

CHAP. IV.

Of the Fluxions of Solids, and of third Fluxions.

PROP. VI.

124. *THE fluxion of the ſolid that is generated by the revolution of the area* ADEF *about the axis* AD, *is meaſured by the cylinder that is generated by the rectangle* EG, *when the fluxion of the axis* AD *is meaſured by* DG.

FIG. 26.

The demonſtration is the ſame as that of the fourth propoſition; and it is eaſy to deduce corollaries from this propoſition ſimilar to thoſe which were inferred from the fourth.

PROP. VII.

125. *The fluxion of a ſolid that can be conceived to be generated by any plane ſurface moving parallel to itſelf, and perpendicular to a given axis, is meaſured by a priſm that has the generating ſurface for its baſe, and its altitude equal to the right line which meaſures the fluxion of the axis.*

By a priſm we underſtand, in this propoſition, any upright ſolid that can be generated by an invariable plane moving parallel to itſelf perpendicular to a given right line. When the generating figure is of an invariable magnitude, the propoſition is demonſtrated as in art. 80. The axis AD being ſuppoſed to flow uniformly,

ly, if the generating plane LM continually increaſe, it is evi- FIG. 34.
dent that the ſolid muſt increaſe with an accelerated motion. Let this motion be ſuppoſed to be continued uniformly from the term when the plane LM cuts the axis in D, and to generate a ſolid equal to R in the ſame time that the axis by flowing uniformly acquires the augment DG. Then ſhall the ſolid R be equal to the priſm upon the baſe LM of the height DG: For, if the ſolid R be ſaid to be greater than that priſm, let AD be produced to P till *lm* (the ſection of the ſolid through P parallel to LM) be to LM as the ſolid R is to that priſm; and the ſolid R ſhall be equal to a priſm upon the baſe *lm* of the height DG. From this it follows, (by theor. 1.) that the motion with which the ſolid EBML flows, continued uniformly, would generate a ſolid equal to a priſm upon the baſe *lm* of the height DP, in the time that the axis acquires the augment DP by flowing uniformly. But the motion with which the ſolid flows when continually accelerated from the ſame term, generates in that time the fruſtum included betwixt the ſections LM and *lm*, which is leſs than the priſm upon the baſe *lm* of the height DP; and, if the motion with which the ſolid flows was continued uniformly from that term, it would generate in the ſame time a leſs ſolid than that fruſtum, (by ax. 1.) and therefore leſs than the priſm upon the baſe *lm* of the height DP. But theſe are contradictory; and therefore the ſolid R is not greater than a priſm upon the baſe LM of the height DG. In like manner it is ſhewn, that the ſolid R is not leſs than this priſm. And the demonſtration is extended to the other caſes of the propoſition in the manner that was deſcribed in art. 90. Therefore, the fluxion of the axis AD being repreſented by DG, the fluxion of the ſolid is accurately meaſured by a priſm upon the baſe LM of the height DG. And, converſely, when the fluxion of the ſolid is repreſented by that priſm, the fluxion of the axis is accurately meaſured by DG.

126. COR. 1. Let the ſolid ALM be a pyramid that has its ver- FIG. 35.
tex in A, and let the ſection LM be triple of the ſquare of AD: Then ſhall the ſolid ALM be always equal to a cube deſcribed upon AD. Therefore the fluxion of the cube upon AD is accurately meaſured by a parallelopipedon upon a baſe that is triple

E. XII. 7. Cor. 1

of

of the ſquare of AD, of a height equal to DG that meaſures the fluxion of the ſide AD, or by a parallelopipedon upon the ſquare of AD of a height triple of DG. From which it follows, that, when four quantities are in continued proportion, and the firſt term is invariable, the fluxion of the ſecond term is to one third part of the fluxion of the fourth as the firſt term is to the third.

127. Cor. II. When the ſolid ALM is a pyramid as in the laſt corollary, and the motion with which AD flows is uniform, the fluxion of the pyramid is as the ſection LM, or as the ſquare of AD. In this caſe, the right lines AL, AM, &c. that form the ſolid angle at A, flow with uniform motions. The ſection LM, and the triangular ſides of the pyramid increaſe with motions that are accelerated uniformly, by prop. 2. cor. 4. But the ſolid ALM increaſes with a motion the acceleration of which increaſes uniformly; that is, the ſecond fluxion of the pyramid increaſes uniformly, and its third fluxion is invariable. In the ſame manner, when the ſide of a cube or the axis of a cone increaſes uniformly, the ſolid flows with a motion the acceleration of which increaſes uniformly.

Fig. 21. 128. In art. 93. we reſolved the trapezium DGHE (which is the increment of the triangle ADE that is generated while AD by increaſing uniformly acquires the augment DG) into two parts, *viz.* the parallelogram EG, which is generated in conſequence of the motion with which the triangle flows at the term when P comes to D, and the triangle EIH, which is generated in conſequence of the acceleration of that motion. In like manner, we are in the preſent caſe to reſolve the increment of the ſolid that is generated while the axis acquires the augment DG, into three parts. The firſt we conceive to be generated in conſequence of the motion with which the ſolid flows, at the term when its ſide or axis becomes equal to AD, ſuppoſed to be continued uniformly for that time. The ſecond part is conceived to be generated in conſequence of the acceleration of this motion ſuppoſed to be continued uniformly from the ſame term and for the ſame time. And the third part is what is generated in conſequence of the continual and uniform increaſe of this acceleration.

Fig. 35.

129. Theſe three parts may be diſtinguiſhed from each other in

In the following manner. Let the triangle MP*m*, by moving perpendicular to the right line AP, generate the pyramid APM*m*. While the point P describes DG, let the pyramid acquire the increment that is terminated by the planes ED*e*, HG*h* parallel to MP*m*; let EI and *ei* parallel to DG meet GH and *gh* in I and *i*, and *e*N parallel to EH meet H*h* in N. Then the frustum of the pyramid terminated by the planes ED*e*, HG*h* is resolved into three parts; the prism ED*e*IG*i*, the prism EIH*ei*N, and the pyramid *e*N*ih*. The first of these, the prism ED*e*IG*i*, measures the first fluxion of the pyramid ADE*e* when DG represents the fluxion of AD, by the seventh proposition. The second, *viz.* the prism EIH*ei*N, measures one half of the fluxion of the first, the altitude DG being supposed invariable. For the fluxion of the triangle ED*e*, or IG*i*, is measured by the parallelogram IN, (the fluxion of DE or GI being measured by IH,) by prop. 2. and the fluxion of the prism ED*e*IG*i* is therefore measured by a parallelopipedon NHE*e* upon the base IN of the altitude DG, which is double of the prism EIH*ei*N that is upon the same base IN and of the same altitude. Therefore the prism EIH*ei*N measures one half of the fluxion of the prism ED*e*IG*i*, or one half of the second fluxion of the pyramid ADE*e*. The last part of the frustum ED*e*HG*h*, *viz.* the pyramid *e*N*ih*, measures one sixth part of the fluxion of the parallelopipedon NHE*e*. For, completing the parallelogram N*i*R*h* and the parallelopipedon N*h*L*e*, the fluxion of the parallelogram IN is measured by the parallelogram NR, (by prop. 1. IH being invariable,) and the fluxion of the parallelopipedon NHE*e* is measured by the parallelopipedon N*h*L*e*, of which the pyramid *e*N*ih* is one sixth part. Therefore this pyramid measures one sixth part of the fluxion of the parallelopipedon NHE*e*, or of the third fluxion of the pyramid ADE*e*. Fig. 36.

130. Thus the solids ED*e*IG*i*, NHE*e*, N*h*L*e* respectively measure the first, second, and third fluxions of the pyramid ADE*e*, when AD flows uniformly and its fluxion is represented by DG. The three parts which constitute the frustum ED*e*HG*h* (or the increment of the pyramid that is generated while AD acquires the augment DG) are, the prism ED*e*IG*i*, the prism EH*ei*N*i*, and the pyramid *e*N*ih*; of which the first measures the first fluxion,

xion, the ſecond meaſures one half of the ſecond fluxion, the laſt meaſures one ſixth part of the third fluxion of the pyramid ADE*e*; and, this part *e*N*ib* being invariable, the pyramid has no fourth fluxion. The firſt of theſe is the ſolid that would have been generated by the motion with which the pyramid ADE*e* flows, if it had been continued uniformly for the time in which P deſcribes DG. The ſecond part is what would have been produced more than the firſt, if the acceleration of the generating motion had been continued uniformly from the term when P comes to D, in the ſame manner as the triangle EIH (fig. 21.) is the ſpace that is generated in conſequence of the acceleration of the motion with which the triangle APM flows while P deſcribes DG, according to what was ſhewn in the 93d article. The third part *e*N*ib* is what is generated in conſequence of the continual uniform increaſe of the acceleration of the generating motion, or of the uniform increaſe of the power by which we may conceive this acceleration to be produced. The firſt fluxion of the pyramid ADE*e* is accurately expreſſed by the firſt of thoſe parts of its increment, and the other two are juſtly neglected in meaſuring it. The ſecond part ſerves for comparing the ſecond fluxion of the pyramid at the term when P comes to D with its ſecond fluxion at any other term, or with the ſecond fluxion of any other ſolid. The third part *e*N*ib* ſerves for comparing the third fluxion of the pyramid with that of any other ſolid, or with the third fluxion of the ſame pyramid when the axis AD flows with a different motion.

131. The priſm EDeIG*i*, which meaſures the firſt fluxion of the pyramid ADE*e*, is to the parallelopipedon NHE*e*, which meaſures its ſecond fluxion, as one half of DE is to EK, or as one half of AD is to DG. The parallelopipedon NHE*e* is to the parallelopipedon N*b*L*e*, which meaſures the third fluxion of the pyramid ADE*e*, as NH is to N*b*, or as AD is to DG. And it is manifeſt, that theſe ratios, by diminiſhing DE, or by diminiſhing the increment of the pyramid EDeHG*h*, may become greater than any aſſignable ratio. Therefore the ratio of the quantity that meaſures the firſt fluxion of the pyramid ADE*e* to that which meaſures its ſecond fluxion, and the ratio of the latter to that which meaſures its third fluxion, may become

greater

greater than any aſſignable ratio, when the quantity that meaſures its firſt fluxion is continually diminiſhed. The pyramid AGH*b* is preciſely equal to the ſum of theſe four ſolids, the pyramid ADE*e*, the priſm EDe IG*i*, the priſm EHIeN*i*, and the pyramid *e*N*ib*. When DG is continually diminiſhed, the firſt of thoſe four ſolids approaches to the pyramid AGH*b*, ſo that their difference may become leſs than any aſſignable magnitude; but the ſum of the firſt two approaches much nearer to the pyramid AGH*b*: and the ratio of the difference betwixt this ſum and that pyramid, to the difference of the pyramids AGH*b* and ADE*e*, may become leſs than any aſſignable ratio. The ſum of the firſt three of thoſe ſolids approaches ſtill nearer to the value of the pyramid AGH*b*; and the ratio of the difference betwixt that ſum and this pyramid, to the difference of the ſum of the firſt two ſolids and the ſame pyramid, may become leſs than any aſſignable ratio. Thus we are led to approximations of various degrees of exactneſs by this method, as well as to accurate menſurations. For, in general, the value of a fluent at the beginning of any ſmall time being given, we approximate to its value at the end of that time, by adding to the former, firſt, the quantity that would have been produced in this time by the generating motion, if it had been continued uniformly from the beginning of the time, (which we ſuppoſe to meaſure the firſt fluxion of the fluent;) then, one half of the quantity that meaſures its ſecond fluxion, and one ſixth part of the quantity that meaſures its third fluxion. The approximation is the more accurate, the more we add of thoſe quantities together, in the order we have deſcribed them: but of this we ſhall treat more fully afterwards.

132. When AD increaſes uniformly, the priſm EDe IG*i* increaſes in the ſame proportion as the triangle EDe, (its altitude DG being invariable,) or in the duplicate ratio of AD; the parallelopipedon NHE*e* increaſes in the ſame proportion as E*e* or AD increaſes, becauſe the ſides HK and HI are invariable; the parallelopipedon N*b*L*e* is always of the ſame magnitude. But when AD is given, and DG, which meaſures the fluxion of AD, increaſes or decreaſes, the priſm EDe IG*i*, which meaſures the firſt fluxion of the pyramid ADE*e*, increaſes or decreaſes in the

the ſame proportion; the parallelopipedon NHE*e*, which meaſures the ſecond fluxion of that pyramid, increaſes or decreaſes in the ſame proportion as its baſe IK, or in the duplicate ratio of DG; and the parallelopipedon N*b*L*e*, which meaſures the third fluxion of the ſame pyramid, increaſes or decreaſes in the triplicate ratio of *ei*, or DG. Therefore, when the firſt fluxion of the pyramid at any given term is ſuppoſed to vary, the ſecond fluxion varies in the duplicate ratio, and the third fluxion in the triplicate ratio of that in which the firſt fluxion varies.

Fig. 6. Pl. 2. 133. All we have ſaid of the pyramid ADE*e* and its fluxions is eaſily transferred to the cone and its fluxions, by what was demonſtrated in the introduction, (pag. 19. & 20.) The increment of the cone AH*b* that is generated while the axis AE is increaſed by EB, or the fruſtum deſcribed by the trapezium EBCH revolving about the axis EB, is equal to the cylinder YX*xy* (BX being a mean proportional betwixt BC and EH) and cone ES*ſ* (BS being the difference of BC and EH) taken together. The cylinder YX*xy* being reſolved into two parts, the cylinder H*Zzb*, and the hollow cylindric ſolid that is generated by the rectangle HX revolving about the axis EB; the firſt of theſe meaſures the firſt fluxion of the cone AH*b*, when the axis flows uniformly and its firſt fluxion is repreſented by EB; and the latter part meaſures one half of the fluxion of the former, or one half of the ſecond fluxion of the cone AH*b*. For, the fluxion of EH being repreſented by CZ, and the fluxion of its ſquare by the rectangle contained by 2EH and CZ, (by prop. 2. cor. 6.) the difference of the ſquares of BX and BZ (or the rectangle contained by EH and CZ) meaſures one half of the fluxion of the ſquare of EH; and the annular ſpace deſcribed by ZX revolving about B meaſures one half of the fluxion of the circle deſcribed by BZ or EH. Therefore the hollow ſolid generated by the rectangle HX meaſures one half of the fluxion of the cylinder H*bz*Z, or one half of the ſecond fluxion of the cone AH*b*. The cone ES*ſ* (which, with the cylinder YX*xy*, completes a ſolid equal to the fruſtum H*bc*C) meaſures one third part of the fluxion of the hollow ſolid deſcribed by the rectangle HX; becauſe the fluxion of the exceſs of the ſquare of BX above the ſquare of BZ (or the fluxion

xion of the rectangle contained by EH and CZ) is measured by the square of BZ or of CS; and the fluxion of the solid generated by the rectangle HX is therefore measured by the cylinder that would be described by the rectangle ES revolving about EB; which cylinder is triple of the cone ES*f*. Thus it appears, that of the three solids which taken together are equal to the frustum H*bc*C, (the increment of the cone AH*b* that is produced while AE is increased by EB,) the cylinder H*bz*Z measures the first fluxion of the cone AH*b*, the hollow solid described by the rectangle HX measures one half of its second fluxion, and the cone ES*f* measures one sixth part of its third fluxion, the axis being supposed to flow uniformly and its fluxion being represented by DG.

134. Let the point *p* describe the right line F*f* in such a manner, that an upright parallelopipedon upon a given invariable base, as the square of any given line *a*, and of a height equal to F*p*, may be always equal to the pyramid APM*m*, (fig. 36.) and, the motion of *p* being then similar to that with which the pyramid flows, the acceleration of its motion shall increase uniformly, or the increments of its velocity generated in any equal times that succeed after each other shall always increase uniformly, when the motion of the point F is uniform. Let *d*S be described by *p* in any given time; let *d*C be the space that would be described in that time by it with its motion continued uniformly from *d*; and let *df* be the space that would have been described in the same time if the acceleration of its motion had been continued uniformly from that term. Then

F *p* *c* f *d* C *f* S m *f*

X *q* V *l* *v* *x* *a*

Z *r* T *t* *z*

shall upright parallelopipedons on the invariable base that is supposed equal to the square of *a*, and of the altitudes *d*C, C*f* and *f*S, be respectively equal to the prisms ED*e*I*Gi*, EH*Ie*N*i* and the pyramid *e*N*ib*; and the right lines *d*C, C*f* and *f*S shall measure the

the firſt fluxion of F*d*, one half of its ſecond fluxion, and one ſixth part of its third fluxion, reſpectively.

135. Let the points *q* and *r* deſcribe the right lines X*x* and Z*z* in ſuch a manner that X*q* may be always equal to the ſpace that would be deſcribed by *p*, and Z*r* may be equal to the ſpace that would be deſcribed by *q*, if their motions were continued uniformly from the terms *p* and *q*, in the ſame time in which it is ſuppoſed that *p* would deſcribe *d*C with its motion continued uniformly from *d*. Then the motion of *q* ſhall be uniformly accelerated, and the motion of *r* uniform, when the motion of P in the line AG (fig. 36.) is ſuppoſed uniform. Let *d*S, V*v* and T*t* be ſpaces deſcribed by the points *p*, *q* and *r* in the ſame time; and let V*l* be the ſpace that would be deſcribed by *q* in that time by its motion continued uniformly from V. Then *d*C, V*l* and T*t* meaſure the firſt, ſecond and third fluxions of F*d*, reſpectively. The upright parallelopipedons on the ſquare of the invariable line *a*, of altitudes reſpectively equal to *d*C, V*l* and T*t* are equal to the ſolids EDeIG*i*, NHE*e* and N*b*L*e*; C*ſ* is one half of V*l*, and *ſ*S is one ſixth part of T*t*; ſo that *d*S is equal to the ſum of *d*C, one half of V*l* and one ſixth part of T*t*, taken together.

136. Becauſe *ſ*S is the difference of *d*S and *d*C, and is equal to one ſixth part of T*t*; it follows, that one ſixth part of the third fluxion of the fluent F*p* is meaſured, at any term, by the difference of the increments that are produced in a given time, when the acceleration of the generating motion is continued uniformly, and when the increaſe of this acceleration is continued uniformly, from that term; and it is determined by a ſimilar difference when the generating motion is continually retarded, or when its acceleration decreaſes.

Fig. 37. 137. The reſt remaining as in the 129th article, let BD be equal to DG, and the plane OB*o* parallel to ED*e* meet AE, A*e*, EI, *ei* and *e*N in the points O, *o*, Y, *y* and *n*. Then, the fruſtum ED*e*OB*o* being equal to the exceſs of the ſum of the priſm ED*e*YB*y* and pyramid *eoyn* above the priſm EY*Oeyn*, it follows, that the difference between the fruſtums ED*e*HG*h* and ED*e*OB*o* is double of the priſm EIH*ei*N, and therefore meaſures the ſecond fluxion of the pyramid ADE*e*, when AD flows uniformly

and

and its fluxion is repreſented by DG. It is alſo manifeſt, that the fruſtum HG*b*OB*o* exceeds the priſm IG*i*YB*y* by a ſolid that is equal to the ſum of the equal pyramids *ei*N*h* and *eony*; and therefore the difference between that fruſtum and priſm meaſures one third part of the third fluxion of the pyramid ADE*e*.

138. In the ſame manner, (fig. art. 134.) if f*d* and *d*S be deſcribed by *p* in equal times, and *dc* be equal to *d*C, then the difference between *d*S and *d*f is equal to 2C*ſ*, or to V*l*; and the difference of fS and *c*C is double of *ſ*S, or equal to one third part of T*t*. The former difference meaſures the ſecond fluxion of F*d*, and the latter meaſures one third part of its third fluxion, when *d*C meaſures its firſt fluxion. From which another theorem may be deduced for determining the third fluxions of quantities. If f*d*, *d*S and Sm be deſcribed by the point *p* in equal times that ſucceed after each other, then it is eaſy to ſee that T*t* is equal to the exceſs of the difference between Sm and *d*S above the difference between *d*S and f*d*. In other caſes, the ratio of T*t* to that exceſs, or of one third of T*t* to the difference between fS and *c*C, may not be a ratio of equality; but it approaches to that ratio, when thoſe increments are continually diminiſhed, as its limit.

139. It appears, from what was ſhewn in the introduction, that the third fluxion of any ſolid that can be generated by a conic ſection revolving on its axis is invariable, when the axis flows uniformly; the parabolic conoid excepted, which has its ſecond fluxion conſtant, and has no third fluxion. This account of third fluxions is ſimilar to that we gave of ſecond fluxions in art. 97. We have deduced thoſe illuſtrations of them from the common geometry, becauſe what relates to ſecond and third fluxions has been repreſented as very abſtruſe, and becauſe it is the application of this part of the method that has been found to be moſt liable to miſtakes.

CHAP.

CHAP. V.

Of the Fluxions of Quantities that are in a continued geometrical Progreſſion, the firſt term of which is invariable.

140. LEt the right lines E*e*, F*f* interſect each other at right angles in A; and, AS, AP and AL being in geometrical progreſſion, let SA be taken upon AF, AP upon AE, and AL upon A*f*; and SPL ſhall be always a right angle. Let SA be invariable; and, while the point P with an uniform motion deſcribes any equal lines p*P*, P*p*, let the point L deſcribe l*L* and L*l*; produce l*p* and *lp* till they meet LP produced in *d* and D. Then, becauſe the angle SP*d* is equal to Sp*d*, the angle PSp is equal to L*d*l. But, ſince SPL is a right angle, the angle SPp is equal to *d*L*l*. Therefore the triangles PSp, L*d*l are ſimilar. In like manner, the triangles PS*p*, LD*l* are ſimilar. Therefore L*l* is to P*p* as DL is to SP, and L*l* is to P*p* as *d*L is to SP; ſo that L*l* is to L*l* as DL is to *d*L. But the angle PSD is equal to D*p*P, or *p*SA, and the angle PS*d* is equal to P*pd*, or pSA; and, conſequently, the angle PSD exceeds PS*d* by DS*d*, equal to *p*Sp: ſo that DL being greater than *d*L, L*l* is greater than L*l*. Therefore, when the motion of P is uniform, the ſpaces deſcribed by L in any equal ſucceeding times, perpetually increaſe, and its motion is accelerated.

Fig. 38.

141. Becauſe the angle *p*SD is equal to *p*PD, or PSA, it is leſs than *p*S*l*, and *p*D is leſs than *pl*. Therefore D*l* is leſs than 2*pl*. But the angle p*Sd* being equal to *d*P*p*, or PSA, it is greater than pSl; and, p*d* being therefore greater than pl, *d*l is greater than 2pl.

142. The velocity of L is to the velocity of P as 2AP is to SA. For, if the ratio of the velocity of L to the velocity of P be a greater ratio than that of 2AP to SA, let it be the ſame as that of 2A*p* (any quantity greater than 2AP) to SA, or of 2*pl* to S*p*. Then, becauſe D*l* is leſs than 2*pl*, the velocity of L ſhall be to the velocity of P in a greater ratio than D*l* is to S*p*,

S*p*, or (the triangles S*p*P, D*l*L being ſimilar) L*l* is to P*p*. But, the motion of L being accelerated, while the motion of P is uniform, it follows from the firſt axiom, that L*l* is greater than the ſpace which would be deſcribed by L if its motion was continued uniformly while P deſcribes P*p*. Therefore the velocity of L is to the velocity of P in a leſs ratio than L*l* is to P*p*. And theſe being contradictory, it follows, that the velocity of L is not to the velocity of P in a ratio greater than that of 2AP to SA. Let the ratio of thoſe velocities be leſs than that of 2AP to SA, and be the ſame as that of 2Ap to SA, or of 2pl to Sp; which being leſs than that of *d*l to Sp, (becauſe *d*l is greater than 2pl,) the velocity of L ſhall be to the velocity of P in a leſs ratio than that of Ll to Pp. But it follows from the ſecond axiom, that Ll is to Pp in a leſs ratio than the velocity of L is to the velocity of P. And theſe being contradictory, it appears, that the velocity of L is to the velocity of P, or the fluxion of AL to the fluxion of AP, as 2AP is to SA.

143. Let the angles SPL, PLM, LMN, MNR, *&c.* be always right, and the angular points P, L, M, N, R, *&c.* deſcribe always the right lines AE, A*f*, A*e*, AF, AE, *&c.* Then ſhall SA, AP, AL, AM, AN, AR, *&c.* be always a ſeries of quantities in a continued geometrical progreſſion, the firſt term of which SA is invariable. While the point P deſcribes any equal lines pP, P*p* on the line AE, let the points L, M, N, R, *&c.* deſcribe the ſpaces lL and L*l*, mM and M*m*, nN and N*n*, rR and R*r*, *&c.* on the lines A*f*, A*e*, AF, AE, the angles Spl and S*pl*, plm and *plm*, lmn and *lmn* being always right. Let lp and *lp* produced meet LP produced in *d* and D; let ml and *ml* produced meet ML in *g* and G; let nm and *nm*, rn and *rn*, *&c.* produced meet NM, RN, *&c.* in *h* and H, *k* and K, *&c.* reſpectively. Then it is manifeſt, that the triangles PSp, L*d*l, M*g*m, N*h*n, *&c.* and the triangles PS*p*, LD*l*, MG*m*, NH*n*, RK*r*, *&c.* are each a ſeries of ſimilar triangles. It is alſo evident, that the angles *p*SA, PSD, LDG, MGH, NHK, *&c.* and the angles pSA, PS*d*, LD*g*, MG*h*, NH*k*, *&c.* are each a ſeries of equal angles; and that any angle of the former ſeries exceeds always the correſponding angle of the latter ſeries by the difference *p*Sp. FIG. 39.

LEMMA VI.

144. *When the first term of a geometrical progression is invariable, and the second term increases uniformly, each of the following terms increases with an accelerated motion.*

Because the triangles PSp, L*d*l, Mgm, N*h*n, R*k*r, &c. are similar, the increments pP, lL, mM, nN, rR, &c. are in the same proportion as the right lines SP, *d*L, *g*M, *h*N, *k*R, &c. and, the triangles PS*p*, LD*l*, MG*m*, NH*n*, &c. being also similar, the increments P*p*, L*l*, M*m*, N*n*, R*r*, &c. are in the same proportion as the right lines SP, DL, GM, HN, KR, &c. Therefore, pP being equal to P*p*, lL is to L*l* as *d*L is to DL, mM is to M*m* as *g*M is to GM, nN is to N*n* as *h*N is to HN, rR is to R*r* as *k*R is to KR. But DL is greater than *d*L, GM is greater than *g*M, HN is greater than *h*N, KR is greater *k*R, and so on; because the angles PSD, LDG, MGH, NHK, &c. exceed the angles PS*d*, LD*g*, MG*h*, NH*k*, &c. respectively, (the excess being always equal to the angle *p*Sp.) Therefore L*l*, M*m*, N*n*, R*r*, &c. exceed Ll, Mm, Nn, Rr, &c. respectively; and, the motion of the point P being uniform, (so that the equal lines pP, P*p* may be described by it in equal times,) the motions of the points L, M, N, R, &c. are perpetually accelerated.

145. It is manifest, that the lines D*l*, G*m*, H*n*, K*r*, &c. are respectively less than 2*pl*, 3*lm*, 4*mn*, 5*nr*, &c. but that the lines *d*l, *g*m, *h*n, *k*r, &c. are respectively greater than 2pl, 3lm, 4mn, 5nr, &c. For, the angles *p*SD, *l*DG, *m*GH, *n*HK, &c. being always equal to each other and to the angle PSA, and the angles *p*S*l*, *lpm*, *mln*, *nmr*, &c. being also equal, the latter always exceed the former by the angle PS*p*. Therefore, as *pl* is greater than *p*D, and D*l* therefore less than 2*pl*; so G*l* is less than 2*lm*, and G*m* less than 3*lm*; H*m* is less than 3*mn*, and H*n* less than 4*mn*; K*n* is less than 4*nr*, and K*r* is less than 5*nr*. In the same manner, the angles pS*d*, l*dg*, m*gh*, n*hk*, &c. being always equal to PSA, and the angles pSl, lpm, mln, nmr, &c. being equal,

Fig. 31

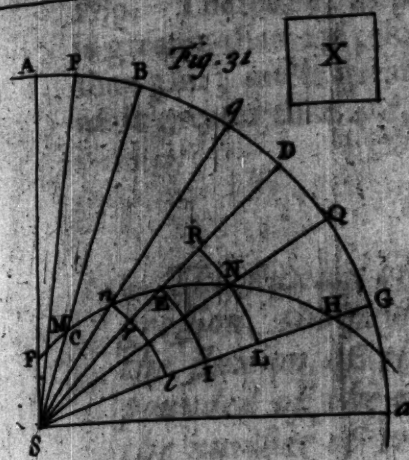


Fig. 32 N. 1.

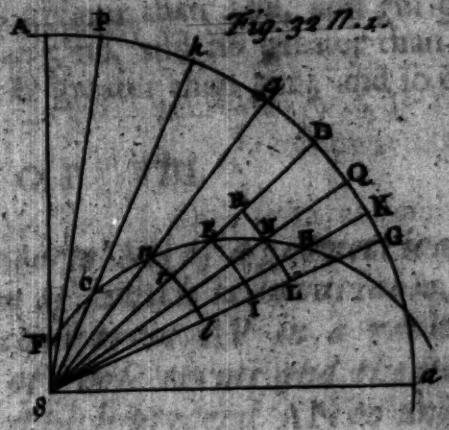


Fig. 32 N. 2.

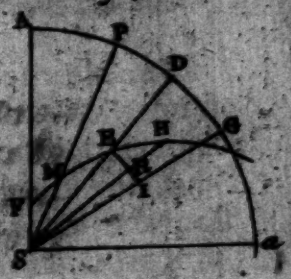


Fig. 33

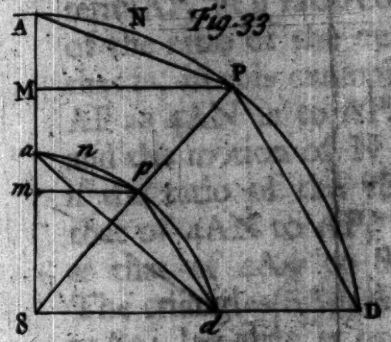


Fig. 34

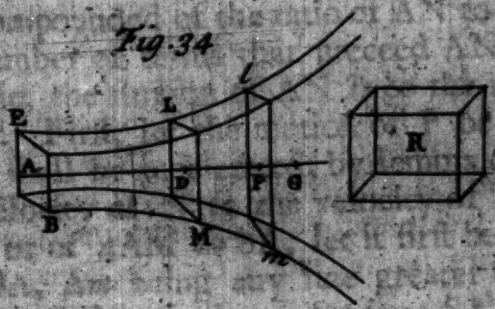


Fig. 35

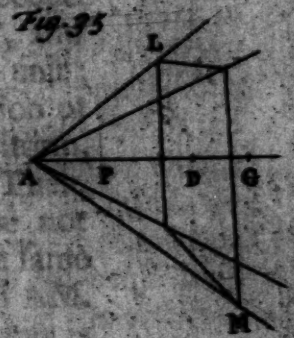


Fig. 36.

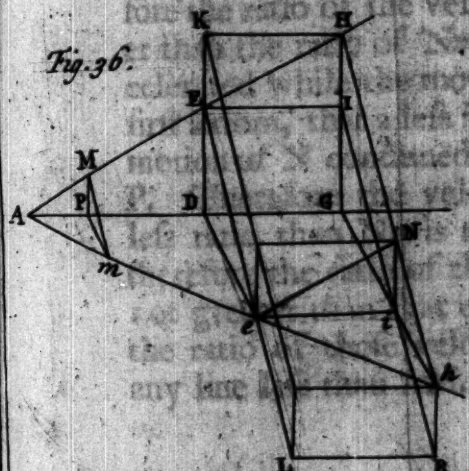


Fig. 37.

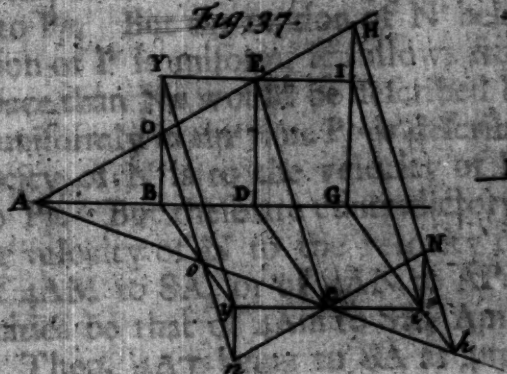
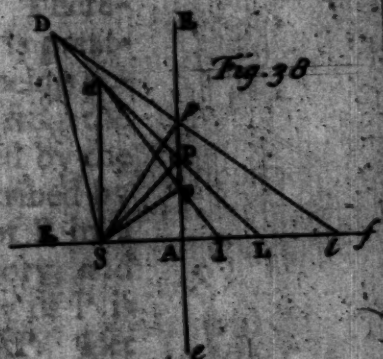
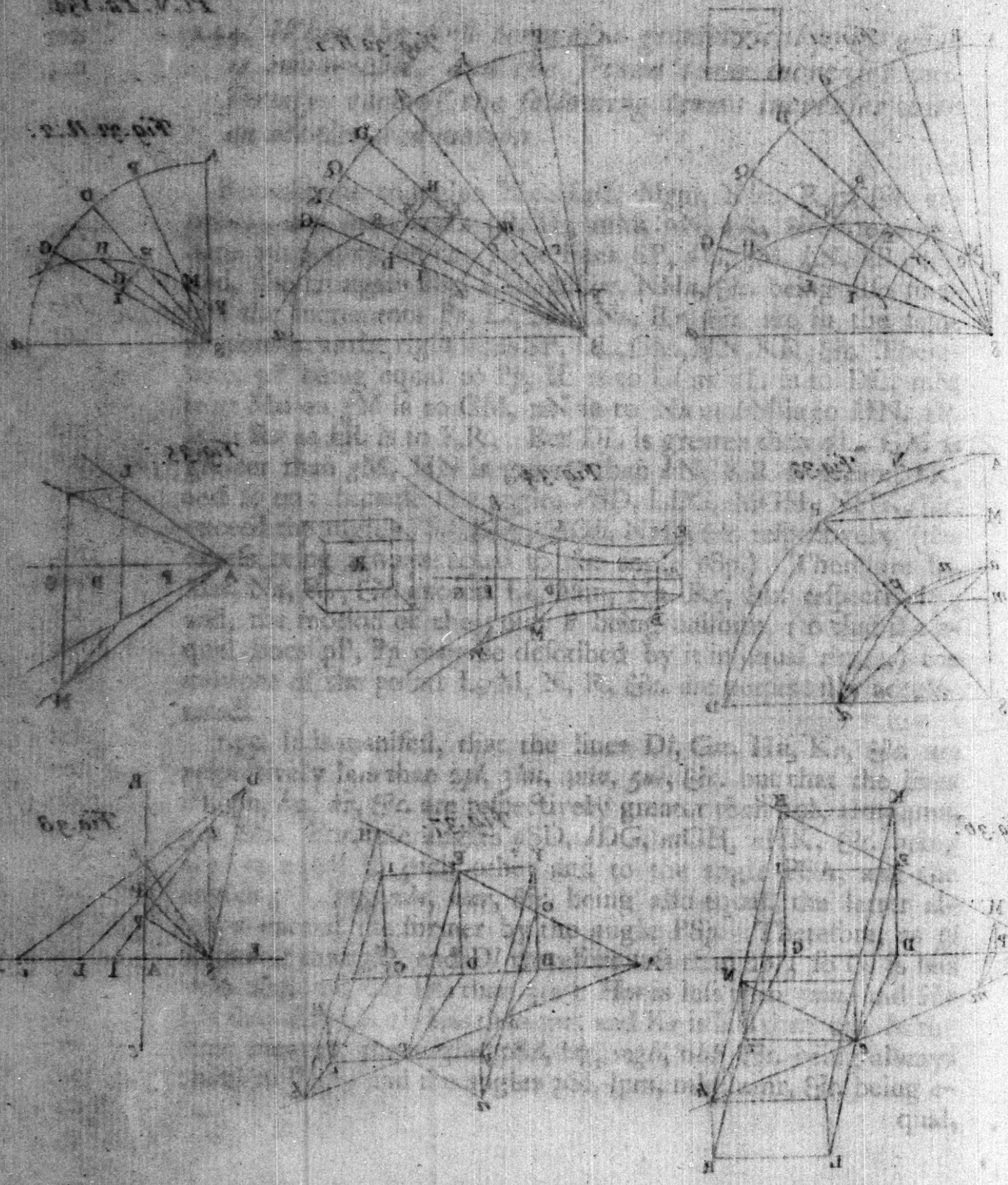


Fig. 38





qual, the former angles always exceed the latter by the angle PSp. Therefore, as p*d* is greater than pl, and conſequently *d*l is greater than 2pl; ſo l*g* is greater than 2lm, and *g*m greater than 3lm; m*h* is greater than 3mn, and n*h* greater than 4nm; n*k* is greater than 4nr, and *k*r greater than 5nr; and ſo on.

PROP. VIII.

146. *The fluxion of any term* AN *of a geometrical progreſſion, the firſt term of which is invariable, is to the fluxion of the ſecond term* AP *in a ratio compounded of the ratio of thoſe terms and the ratio of the number of terms which preceed* AN *to unit.*

The fluxion of any term AN is to the fluxion of the ſecond term AP, in a ratio compounded of the ratio of AN to AP, and of the ratio of the number of terms that preceed AN to unit; that is, in this example, the fluxion of AN is to the fluxion of AP as 4AN is to AP. Firſt, let the motion of P be uniform, and the motion of N ſhall be accelerated, by lemma 6. Then, if the ratio of the velocity of N to the velocity of P be not that of 4AN to AP, or of 4AM to SA, let it firſt be the ſame as that of 4A*m* to SA, A*m* being any line greater than AM. The triangles SA*p*, A*mn* being ſimilar, 4A*m* is to SA as 4*mn* is to S*p*; but 4*mn* is greater than H*n*, (by art. 145.) and 4*mn* is to S*p* in a greater ratio than H*n* is to S*p*, or N*n* to P*p*. Therefore the ratio of the velocity of N to the velocity of P is greater than the ratio of N*n* to P*p*. But the motion of N being accelerated while the motion of P is uniform, it follows from the firſt axiom, that a leſs ſpace than N*n* would be deſcribed by the motion of N continued uniformly in the time P*p* is deſcribed by P. Therefore the velocity of N is to the velocity of P in a leſs ratio than N*n* is to P*p*. But theſe are contradictory, and therefore the ratio of the velocity of N to the velocity of P is not greater than that of 4AM to SA, or of 4AN to AP. Let the ratio of thoſe velocities be that of 4Am to SA, Am being any line leſs than AM. Then, 4Am being to SA as 4mn is to

Sp, and 4mn being leſs than *bn*, (by art. 145.) it follows, that the velocity of N is to the velocity of P in a leſs ratio than *bn* is to Sp, or Nn is to Pp. But, by the ſecond axiom, a greater ſpace than Nn would be deſcribed by the motion of N continued uniformly, in the time pP is deſcribed by P; and therefore the velocity of N is to the velocity of P in a greater ratio than Nn is to Pp. But theſe being alſo contradictory, the velocity of N is to the velocity of P neither in a greater nor in a leſs ratio than that of 4AM to SA, or of 4AN to AP; and therefore the fluxion of AN is to the fluxion of AP preciſely in the ſame ratio as 4AN is to AP. All the other caſes of this propoſition, when the motion of P is variable, are eaſily deduced from this caſe, by the eleventh general theorem. And it is obvious, that this demonſtration is applicable in the ſame manner to any other terms of the progreſſion.

PROP. IX.

147. *The fluxions of any two terms in a geometrical progreſſion, the firſt term of which is invariable, are in a ratio compounded of the ratio of thoſe terms to each other, and of the ratio of the numbers that ſhew how many terms preceed them in the progreſſion.*

This follows from the laſt propoſition. The fluxion of AN, for example, is to the fluxion of AM as 4AN is to 3AM. For the fluxion of AN is to the fluxion of AP as 4AN is to AP, by the laſt propoſition; and the fluxion of AM is to the fluxion of AP as 3AM is to AP, by the ſame. Therefore the fluxion of AN is to the fluxion of AM as 4AN is to 3AM; and it is manifeſt, that the ſame demonſtration may be applied when the fluxions of any other terms of the progreſſion are compared together.

148. It follows from what has been demonſtrated, that, if the motions of the points P, L, M, N, R, *&c.* were continued uniformly, lines reſpectively equal to SA, 2AP, 3AL, 4AM, 5AR, *&c.* would be deſcribed by them in the ſame time.

149. The

149. The ſecond, third, and higher fluxions of any term of the progreſſion, may be repreſented by certain multiples of the preceeding terms when the firſt fluxion of the ſecond term AP is conſtant (or the motion of P is uniform), and is repreſented by the invariable right line SA. For, the velocity of the point N, or the firſt fluxion of AN, being repreſented by 4AM; the fluxion of this velocity, or the ſecond fluxion of AN, may be repreſented by four times the fluxion of AM, (which is itſelf meaſured by 3AL,) or by 12AL; the third fluxion of AN is repreſented by 24AP, (becauſe the fluxion of AL is repreſented by 2AP,) its fourth fluxion by 24SA, and it has no fifth fluxion.

150. If A*u* be a mean proportional between AL and A*l*, and AP flow uniformly, then is 2L*u* equal to the line that would be generated by the motion with which AL flows continued uniformly for the time in which P deſcribes P*p*; and the difference betwixt *lu* and L*u* is what is generated in conſequence of the acceleration of this motion during that time. For A*u* is to AL in the ſubduplicate ratio of A*l* to AL, or in the ratio of A*p* to AP; and L*u* is to P*p* as AL is to AP, or as AP is to SA. Therefore 2L*u* is to P*p* as 2AP is to SA, or as the velocity of the point L is to the velocity of P. Let A*b* and A*k* be two mean proportionals between AM and A*m*; and, if the motion of M was continued uniformly while P deſcribes P*p*, a line would be deſcribed by it equal to 3M*b*; if the acceleration of that motion was continued uniformly from the ſame term and for the ſame time, a line would be deſcribed by it equal to 3*bk*; and what would be generated in conſequence of this uniform acceleration is equal to thrice the difference betwixt *bk* and M*b*. The difference betwixt M*m* and 3*bk*, or the exceſs of the difference betwixt *km* and *bk* above the difference betwixt *bk* and M*b*, is what is generated in conſequence of the increaſe of that acceleration during the ſame time. The proportions betwixt thoſe differences and the lines that meaſure the firſt, ſecond and third fluxions of AM, eaſily appear, and have been already ſhewn in the laſt chapter. Any term has fluxions of as many degrees as there are terms that preceed it in the progreſſion, when the fluxion of AP is invariable; and the increment of any term that is generated in a given time, may be reſolved in this manner into

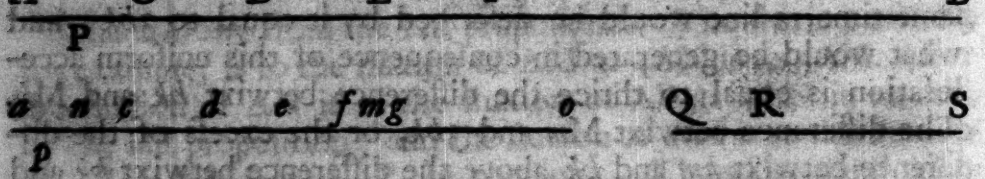
FIG. 40.

to as many parts as it has fluxions of different orders; and each part may be conceived to be generated in consequence of its respective fluxion. But this will appear more easily afterwards.

CHAP. VI.

Of Logarithms, and the Fluxions of logarithmic Quantities.

151. THE Logarithms were invented for facilitating computations in Arithmetic and Trigonometry, by the celebrated Lord NAPIER, Baron of Merchiston, and have been found since to be of great use in the higher Geometry, particularly in the method of Fluxions. Their nature and genesis is proposed by the inventor in a method similar to that which is applied in this doctrine for explaining the genesis of quantities of all sorts, and is described by him almost in the same terms. He begins his treatise on this subject, by defining that a line increases equally, when the point that describes it moves over equal spaces in equal times. Let A (says he *) be the term from which the line is to be described by the flux or motion of



the point P. Let it flow from A to C in the first moment, (or in any small part of the time,) from C to D in the second mo-

* *Sit punctus* A *à quo ducenda sit linea fluxu alterius puncti qui sit* P. *Fluat ergo primo momento* P *ab* A *in* C, *secundo momento à* C *in* D, &c. Mirif. Logar. Canon. descript. defin. 1. He afterwards lays down a postulate, similar to what we assumed in the first chapter, in these words: *Quum quolibet motu & tardior & velocior dari possit, sequetur necessariò cuique motui æquivelocem (quem nec tardiorem nec velociorem definimus) dari posse.*

ment,

ment, from D to E in the third; and ſo on for ever, deſcribing always the equal parts AC, CD, DE, EF, &c. in equal times. This line is then ſaid to increaſe *equally*.

152. By his ſecond definition *, a line decreaſes *proportionally*, when the point that moves over it deſcribes ſuch parts in equal times as are always in the ſame conſtant ratio to the lines from which they are ſubducted, or to the diſtances of that point at the beginning of thoſe times from a given term in the line. Let the ratio of QR to QS be any given ratio; let *ac* be to *ao*, *cd* to *co*, *de* to *do*, *ef* to *eo*, *fg* to *fo*, &c. always in the ſame invariable ratio of QR to QS. Suppoſe that the point *p* ſets out from *a*, deſcribing *ac*, *cd*, *de*, *ef*, *fg*, &c. in equal parts of the time; and let the ſpace deſcribed by *p* in any given time be always in the ſame ratio to the diſtance of *p* from *o* at the beginning of that time. Then the right line *po* is ſaid to decreaſe proportionally. It is manifeſt, that the lines *ao*, *co*, *do*, *eo*, *fo*, &c. or the diſtances of the point *p* from *o* at equal ſucceeding intervals of time, are in a continued geometrical progreſſion.

153. Suppoſe now that the uniform motion of the point P in deſcribing the line AB is equal to the motion with which *p* ſets out from *a* in deſcribing the line *ao*; and the line AP that is deſcribed by P with this uniform motion, in the ſame time that *oa* by decreaſing proportionally becomes equal to *op*, is the *Logarithm* of *op* †. Thus AC, AD, AE, AF, &c. are the logarithms of *oc*, *od*, *oe*, *of*, &c. reſpectively; and *oa* is the quantity whoſe logarithm is ſuppoſed equal to nothing.

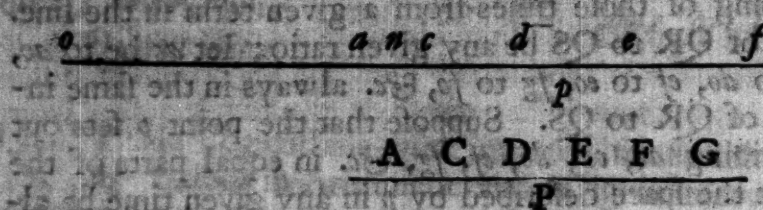
154. In like manner, ſuppoſe the line *oa* to increaſe proportionally; that is, let the point *p* move in the line *oa* produced beyond *a*, and in any equal times deſcribe ſpaces proportional to its diſtances from *o* at the beginning of each time; ſo that

* *Linea proportionaliter in breviorem decreſcere dicitur, quum punctus eam tranſcurrens æqualibus momentis ſegmenta abſcindit ejuſdem continuò rationis ad lineas à quibus abſcinduntur.* Ibid. defin. 2.

† *Logarithmus cujuſque ſinûs eſt numerus quàm proximè definiens lineam quæ æqualiter crevit, interea dum ſinûs totius linea proportionaliter in illum ſinum decrevit, exiſtente utroque motu ſynchrono, atque initio æquivelocè,* Ibid. That this doctrine may be more general, we abſtract from numbers, and repreſent the logarithms by lines, as well as the quantities of which they are the logarithms.

the

the increments *ac*, *cd*, *de*, *ef*, &c. may be described by it in equal times, when *ac* is to *ao*, *cd* to *co*, *de* to *do*, *ef* to *eo*, &c. always in the same invariable ratio. When *p* sets out from *a*, let P set out from A with an equal velocity; then, the motion of P being continued uniformly while *op* increases proportionally, the line AP is always the logarithm of *op*; and the lines AC, AD, AE, AF, &c. are the logarithms of *oc*, *od*, *oe*, *of*, &c. re-



spectively. It is manifest, that the lines *oa*, *oc*, *od*, *oe*, *of*, &c. are in continued geometrical proportion. If *ac* and *fg* be described by *p* in equal intervals of time, and *an*, *fm* be described by it in any equal parts of those times; then shall *ac* be to *fg*, and *an* to *fm*, as *oa* is to *of*, or as *oc* is to *og*, or as *on* is to *om*; that is, the spaces described by *p* in equal times are in the same proportion to each other as the distances of *p* from *o*, at the beginning, end, or any similar term of those times.

155. When a ratio is given, the point *p* describes the difference of the terms of the ratio in the same time. When a ratio is duplicate of another ratio, the point *p* describes the difference of the terms in a double time. When a ratio is triplicate of another, it describes the difference of the terms in a triple time; and so on. Thus the ratio of *od* to *oa* is duplicate of the ratio of *oc* to *oa*; and, the lines *ac*, *cd* being described by *p* in equal times, the time in which *ad* is described by it is double of that in which it describes *ac*. The ratio of *oe* to *oa* is triplicate of the ratio of *oc* to *oa*, and the time in which *ae* is described by *p* is triple of that in which it describes *ac*. In the same manner it appears, that, when a ratio is compounded of two or more ratios, the point *p* describes the difference of the terms of that ratio, in a time equal to the sum of the times in which it describes the differences of the terms of the simple ratios of which it is compounded.

156. What

156. What we have ſaid of the times of the motion of *p* when *op* increaſes proportionally, is to be applied to the ſpaces that are deſcribed by P in thoſe times with its uniform motion; and hence the chief properties of the logarithms are deduced. The difference of the logarithms of the terms of the ſame ratio is always the ſame. When a ratio is duplicate of another ratio, the difference of the logarithms of the terms is double; when a ratio is triplicate, that difference is triple; and ſo on. The difference of the logarithms of the terms of a compound ratio, is equal to the ſum that is produced by adding together the differences of the logarithms of the terms of the ſimple ratios from which it is compounded.

157. Thus logarithms are the meaſures of ratios. The exceſs of the logarithm of the antecedent above the logarithm of the conſequent meaſures the ratio of thoſe terms. The meaſure of the ratio of a greater quantity to a leſſer is poſitive, as this ratio compounded with any other ratio increaſes it. The ratio of equality compounded with any other ratio neither increaſes nor diminiſhes it; and its meaſure is nothing. The meaſure of the ratio of a leſſer quantity to a greater is negative, as this ratio compounded with any other ratio diminiſhes it. The ratio of any quantity A to unit compounded with the ratio of unit to A produces the ratio of A to A, or the ratio of equality; and the meaſures of thoſe two ratios deſtroy each other when added together; ſo that, when the one is conſidered as poſitive, the other is to be conſidered as negative. By ſuppoſing the logarithms of quantities greater than *oa* (which is ſuppoſed to repreſent unit) to be poſitive, and the logarithms of quantities leſs than it to be negative, the ſame rules ſerve for the operations by logarithms, whether the quantities be greater or leſs than *oa*. From what we have ſaid, it is eaſy to ſee how the logarithms ſerve for abridging computations. Becauſe unit is to any quantity A as B is to the product of A and B, and the logarithm of unit is nothing, the logarithm of any product is equal to the ſum of the logarithms of the factors, and the logarithm of any power of A is to the logarithm of A as the exponent of that power is to unit.

158. When *op* increaſes proportionally, the motion of *p* is

perpetually accelerated; for the ſpaces *ac*, *cd*, *de*, &c. that are deſcribed by it in any equal times that continually ſucceed after each other, perpetually increaſe in the ſame proportion as the lines *oa*, *oc*, *od*, &c. When the point *p* moves from *a* towards *o* as in the 152d article, and *op* decreaſes proportionally, the motion of *p* is perpetually retarded; for the ſpaces deſcribed by it in any equal times that continually ſucceed after each other, decreaſe in this caſe in the ſame proportion as *op* decreaſes.

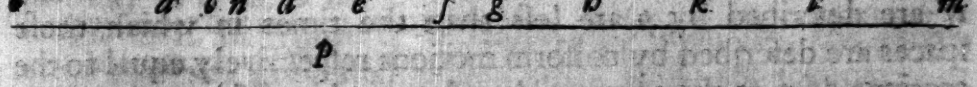
PROP. X.

159. *The fluxion of a quantity that increaſes or decreaſes* proportionally, *varies always in the ſame ratio as the quantity itſelf.*

Let the line *op* increaſe or decreaſe proportionally, and the velocity of the point *p* (or the fluxion of *op*) ſhall always vary in the ſame ratio as its diſtance from *o*. Thus the velocity of *p* at the term *c* ſhall be to its velocity at any other term *g*, as *oc* is to *og*. For, if the ratio of the velocity of *p* at *g* to its velocity at *c* be greater than that of *og* to *oc*, let it be the ſame as that of any line *oh* greater than *og* to *oc*; and let *ok* be to *oc* as *og* is to *oh*. Then is *gh* to *kc* as *oh* is to *oc*, or as *og* is to *ok*; and the lines *kc*, *gh* are deſcribed by the point *p* in the ſame time, by the ſuppoſition. Let *op* firſt increaſe proportionally: and, becauſe the motion of *p* is then perpetually accelerated, (art. 158.) it follows from the firſt axiom, that a leſs line than *gh* would be deſcribed by it, if its motion was continued uniformly from *g*, in the ſame time that it deſcribes *gh* while *op* increaſes proportionally; and it follows from the ſecond axiom, that a greater line than *kc* would be deſcribed in an equal time by *p*, if its motion was continued uniformly from the term *c*. Therefore the velocity of *p* at *g* is to its velocity at *c* in a ratio that is leſs than the ratio of *gh* to *kc*, or of *oh* to *oc*. But it was ſuppoſed, that the velocity of *p* at *g* is to the velocity

locity of *p* at *c* as *oh* is to *oc*. And theſe being contradictory, it follows, that the velocity of *p* at *g* is not to its velocity at *c* in any ratio greater than that of *og* to *oc*. In the ſame manner it appears, that the velocity of *p* at *c* is not to its velocity at *g* in any ratio greater than that of *oc* to *og*; from which it follows, that the velocity of *p* at *g* is not to the velocity of *p* at *c* in any ratio that is leſs than the ratio of *og* to *oc*. Therefore the velocity of *p* at *g* is to the velocity of *p* at *c* preciſely as *og* is to *oc*; and the fluxion of *og* is to the fluxion of *oc* in the ſame ratio. When *op* decreaſes proportionally, the demonſtration is deduced in the ſame manner from the third and fourth axioms.

160. But it may be of uſe for illuſtrating the account of logarithms that is propoſed by the inventor, to demonſtrate the converſe of this propoſition from the axioms, and ſhew that when the velocity of *p* is always as the diſtance *op*, then this line increaſes or decreaſes in the manner that is ſuppoſed by him. Let the velocity of *p* therefore increaſe always in the ſame proportion as *op* increaſes; let *af* be to *gm* as *oa* is to *og*: and the ſpaces *af*, *gm* ſhall be deſcribed by the point *p* in equal times; that is, *op* ſhall increaſe proportionally. For, if the time in which *af* is deſcribed by *p* be ſuppoſed to be greater than the time in which *gm* is deſcribed by it, let *on* be greater than *oa* in the ſame ratio. Let *oa*, *oc*, *od*, *oe*, *of* be a ſeries of lines in continued proportion, the number of proportionals between *oa* and *of* being increaſed till the term *oc* (which is next to *oa*) become leſs than *on*. Becauſe *oa* is to *of* as *og* is to *om*, if *og*, *oh*,



ok, *ol*, *om* be alſo in continued proportion, and the number of terms be the ſame as in the other ſeries, the lines *ac*, *cd*, *de*, *ef*, *gh*, *hk*, *kl*, *lm* ſhall be in the ſame proportion to each other reſpectively, as the lines *oa*, *oc*, *od*, *oe*, *og*, *oh*, *ok*, *ol*, or as the lines *oc*, *od*, *oe*, *of*, *oh*, *ok*, *ol*, *om*. Suppoſe firſt the line *ac* to be deſcribed by an uniform motion equal to that of *p* at the term *a*, *cd* to be deſcribed by an uniform motion equal to that of *p* at

the term *c*; and *de*, *ef*, *gh*, *hk*, *kl*, *lm* to be deſcribed by uniform motions equal to thoſe of the point *p* at the reſpective terms *d*, *e*, *g*, *h*, *k*, *l*. Then, theſe ſpaces *ac*, *cd*, *de*, *ef*, *gh*, *hk*, *kl*, *lm* being in the ſame ratios to each other as the velocities of the uniform motions by which they are ſuppoſed to be deſcribed, it follows, that they are all deſcribed in this caſe in equal times; and, the times in which *af* and *gm* are thus deſcribed being equimultiples of the times in which *ac* and *gh* are deſcribed, they are therefore equal. Suppoſe now the ſame ſpaces *ac*, *cd*, *de*, *ef*, *gh*, *hk*, *kl*, *lm* to be deſcribed by uniform motions reſpectively equal to thoſe of *p* at the terms *c*, *d*, *e*, *f*, *h*, *k*, *l*, *m*, and the times in which they are in this caſe deſcribed, are all equal to each other, for the ſame reaſon; and, conſequently, the times in which *af* and *gm* are thus deſcribed are equal in this caſe alſo. The time in which *af* is deſcribed in the firſt of thoſe two caſes, is to the time in which it is deſcribed in the ſecond, as the time in which the line *ac* is deſcribed in the firſt is to the time in which the ſame line is deſcribed in the ſecond, (Elem. 15. 5.) or as the velocity of the uniform motion by which it is deſcribed in the ſecond is to the velocity of the uniform motion by which it is deſcribed in the firſt caſe; that is, as *oc* is to *oa*. From which it follows, that the time in which *af* is deſcribed in the firſt caſe is to the time in which *gm* is deſcribed in the ſecond (which is equal to the time in which *af* is deſcribed in the ſame caſe) in the ſame ratio of *oc* to *oa*. But, the motion of *p* being accelerated continually while *op* increaſes, the time in which *ac* is deſcribed by *p* is leſs than the time in which *ac* is deſcribed by an uniform motion equal to that of *p* at *a* according to the firſt ſuppoſition. In the ſame manner, the times in which *cd*, *de* and *ef* are deſcribed by *p* are leſs than the times in which thoſe ſpaces are deſcribed by uniform motions reſpectively equal to the motions of *p* at the terms *c*, *d*, and *e*; and, conſequently, *af* is deſcribed by *p* in a time leſs than that in which *af* is deſcribed in the firſt of the two caſes that were ſuppoſed by us. In the ſame manner, the time in which *gh* is deſcribed by *p* is greater than that in which *gh* is deſcribed by an uniform motion equal to that of *p* at *h*; and the time in which *gm* is deſcribed by *p* is greater than the time in which *gm* is deſcribed in the ſecond

of

of thoſe two caſes. Therefore the time in which *af* is deſcribed by *p* is to the time in which *gm* is deſcribed by it, in a leſs ratio than the time in which *af* is deſcribed in the former caſe is to the time in which *gm* is deſcribed in the latter; that is, in a leſs ratio than *oc* is to *oa*; and, conſequently, in a leſs ratio than *on* is to *oa*. But the times in which *af* and *gm* are deſcribed by *p* were ſuppoſed to be in the ſame ratio as *on* is to *oa*. And theſe being contradictory, it follows, that the time in which *af* is deſcribed by *p* is not greater than the time in which *gm* is deſcribed by it. In the ſame manner it appears, that the latter of thoſe times cannot exceed the former. Therefore the lines *af* and *gm* are deſcribed by *p* in equal times when the velocity of *p* increaſes in the ſame proportion as *op* increaſes. A demonſtration of the ſame kind is eaſily adapted to the caſe when *op* is ſuppoſed to decreaſe.

161. Cor. I. The fluxion of any quantity *op* is to the fluxion of its logarithm AP (fig. art. 151. & 154.) as *op* is to *oa*, which repreſents unit and has its logarithm equal to nothing. The motion of P, by which AP the logarithm of *op* is deſcribed, was ſuppoſed (in art. 153. & 154.) uniform, and equal to the velocity of *p* at *a*, *oa* being the quantity whoſe logarithm is nothing. Therefore, by this propoſition, the velocity of *p* at any term of the time is to the conſtant velocity of P as *op* is to *oa*. When the logarithm AP is ſuppoſed to flow with a variable motion, *op* does not increaſe proportionally: but, from what we have ſhewn, it may be demonſtrated by the eleventh general theorem, that the velocity of *p* is ſtill to the velocity of P (*op* being always the line of which AP is the logarithm) as *op* is to *oa*.

162. Cor. II. When *op* increaſes proportionally, the increments that are generated in any equal times are accurately in the ſame ratio as the velocities of *p*, or the fluxions of *op*, at the beginning, end, or at any ſimilar terms of thoſe times. Thus *af* is to *gm* (fig. art. 160.) as *oa* is to *og*, or as *of* is to *om*, or as *oc* is to *oh*.

163. Cor. III. When *op* increaſes or decreaſes proportionally, the fluxions of this line of all the higher orders increaſe or decreaſe in the ſame proportion as the line itſelf increaſes or decreaſes;

creaſes; ſo that one rule ſerves for comparing together thoſe of any kind at different terms of the time. For, the velocity of p being always as op, it flows in the ſame manner as op flows, and its fluxion (or the ſecond fluxion of op) varies in the ſame manner as the firſt fluxion of op, or as the fluent op itſelf varies. In the ſame manner, when the ſecond fluxion of op is conſidered as a flowing quantity, its fluxion (or the third fluxion of op) is as the velocity of p, or as the line op. It is evident, that all the higher fluxions of op vary in the ſame manner; and that in this caſe we never arrive at any conſtant or invariable fluxion. In treating of thoſe higher orders of fluxions, we never ſuppoſe any fluxion of op to be the velocity of a velocity; but we find, that the fluxion of any order flows in the ſame manner as op flows; and, conſidering that fluxion as a flowing quantity, its fluxion is as the velocity of p or as the line op. The firſt fluxion of any fluent is not the velocity of that fluent, but the velocity of the motion by which it is conceived to be generated; and, in like manner, the ſecond fluxion of that fluent is not the velocity of the velocity of this fluent, but the velocity of the motion by which the quantity is generated that always repreſents its firſt fluxion: but of this we ſaid enough above, in the firſt chapter, from art. 70. to the end, and in art. 97. 134. 135. 137. & 138.

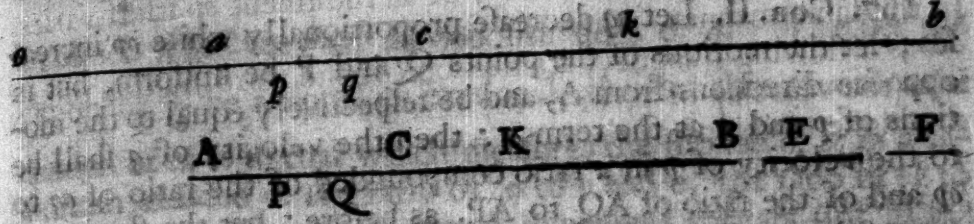
PROP. XI.

165. *Let the logarithms of two quantities be always to each other in any invariable ratio, and the fluxions of thoſe quantities ſhall be in a ratio that is compounded of the ratio of the quantities themſelves and of the invariable ratio of their logarithms.*

Let the points Q and P ſet out from A together, and deſcribe AB with uniform motions; and let the velocity of Q be to the velocity of P in the invariable ratio of E to F. Let the points q and p ſet out at the ſame time from a in the right line ab with velocities reſpectively equal to the velocities of Q and P. Then,

if

if *oq* and *op* increaſe proportionally, AQ ſhall be the logarithm of *oq*, and AP the logarithm of *op*; and, AQ being to AP as the velocity of Q is to the velocity of P, or as E is to F, it follows, that the logarithm of *oq* is always to the logarithm of



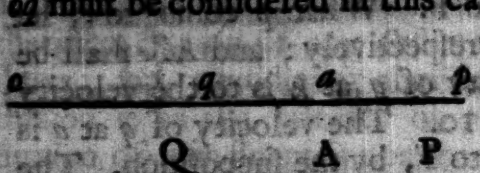
op in the invariable ratio of E to F. When *p* comes to *c*, let P, *q* and Q come to C, *k* and K reſpectively; and AK ſhall be to AC as E is to F. The velocity of *q* at *k* is to the velocity of *q* at *a* as *ok* is to *oa*, by prop. 10. The velocity of *q* at *a* is to the velocity of *p* at *a* as E is to F, by the ſuppoſition. The velocity of *p* at *a* is to the velocity of *p* at *c* as *oa* is to *oc*, by prop. 10. Therefore, by compounding thoſe ratios together, the velocity of *q* at *k* is to the velocity of *p* at *c* as the rectangle contained by *ok* and E is to the rectangle contained by *oc* and F. But the fluxion of *ok* is to the fluxion of *oc* as the velocity of *q* at *k* is to the velocity of *p* at *c*; that is, in a ratio compounded of that of the lines *ok* and *oc* and of the invariable ratio of their logarithms AK and AC.

166. Cor. I. When *oa*, *oc* and *ok* are in continued geometrical progreſſion, the logarithm of *ok* is always double of the logarithm of *oc*, by art. 156. and therefore the fluxion of *ok* is to the fluxion of *oc* as 2*ok* is to *oc*, or as 2*oc* is to *oa*, as has been already demonſtrated in the 96th and 142d articles. When *ok* is any term of a continued geometrical progreſſion of which *oa* and *oc* are the two firſt terms, and *ok* occupies always the ſame place in this progreſſion; then, becauſe the logarithm of *ok* is to the logarithm of *oc* as the number of terms that preceed *ok* in the progreſſion is to unit, the fluxion of *ok* is to the fluxion of *oc* in a ratio compounded of the ratios of *ok* to *oc* and of the number of terms that preceed *ok* to unit, as was demonſtrated in the eighth propoſition. In general, when *oc* is any other term of the progreſſion, the logarithm of *ok* is to the logarithm of *oc*

as

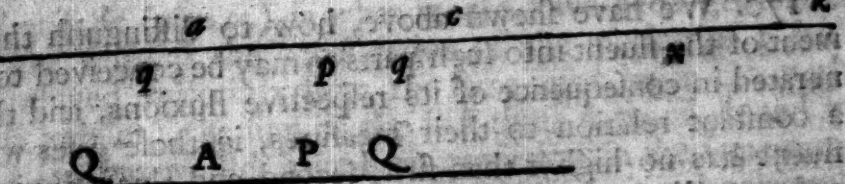
as the number of terms that preceed *ok* is to the number of terms that preceed *oc*; and the fluxion of *ok* is to the fluxion of *oc* in a ratio compounded of the ratio of those numbers and of the ratio of *ok* to *oc*, as we demonstrated in the ninth proposition in a different manner.

167. Cor. II. Let *oq* decrease proportionally while *op* increases; let the motions of the points Q and P be uniform, but in opposite directions from A, and be respectively equal to the motions of *p* and *q* at the term *a*: then the velocity of *q* shall be to the velocity of *p* in a ratio compounded of the ratio of *oq* to *op* and of the ratio of AQ to AP, as before: but the fluxion of *oq* must be considered in this case as negative, because *oq* decreases. When *op*, *oa* and *oq* are in continued proportion, AQ the logarithm of *oq* is equal to AP the logarithm of *op*, the velocity of *q* is to the velocity of *p* as *oq* is to *op*; and any lines that measure the fluxions of the terms *oq* and *op* are in the same ratio as those terms themselves. In general, when *oq*, *oa* and *op* are terms of any geometrical progression, and the number of terms from *oa* to *oq* (including one of those terms themselves) is to the number of terms from *oa* to *op* (including one of these) as *m* is to *n*; then AQ is to AP as *m* is to *n*; and the velocity of *q* is to the velocity of *p* in a ratio compounded of the ratios of *oq* to *op* and of *m* to *n*. Any quantities that measure the fluxions of *oq* and *op* are in the same ratio; but the fluxion of *oq* is considered as negative when the fluxion of *op* is positive. Thus it appears how the fluxions of the terms of a geometrical progression are determined when any intermediate term of the progression is invariable.



168. Cor. III. In those cases, the velocity of Q is to the velocity of P, or the logarithm of *oq* to the logarithm of *op*, in a ratio which can be expressed by that of rational numbers. But the proposition is general, and the ratio of the fluxions of *oq* and *op* is assigned by it when the ratio of AQ to AP is invariable, though this ratio cannot be expressed by that of number to number. According to the notation which is now in use in Algebra, when AQ is to AP as *m* is to *n*, and *oa* being unit, *oc* is expressed

fed by A, and it is ſuppoſed (as in art. 165.) that *oq* becomes equal to *ok* when *op* becomes equal to *oc*, *ok* is expreſſed by $A^{\frac{m}{n}}$ when the points *p* and *q* are on the ſame ſide of the point *o*, and by $A^{-\frac{m}{n}}$ when they are on oppoſite ſides of *o*. Whether the exponent $\frac{m}{n}$ be rational or irrational, the line that meaſures the fluxion of *ok* is to that which meaſures the fluxion of *oc* in a ratio compounded of the ratio of *m* to *n* and of the ratio of *ok* to *oc*. This notation by irrational exponents appears firſt in Sir Isaac Newton's letter to Mr. Oldenburgh of October 24. 1676. If the ratio of AQ and AP, the logarithms of *oq* and *op*, be variable, let it be the ſame as that of *ox* to *oa*, (or unit;)



and, *op* being expreſſed by A, *ox* by *x*, *oq* may be expreſſed by A^x, (the logarithms of powers and their exponents being always in the ſame ratio,) and is called an *exponential* quantity of the firſt degree by Mr. Leibnitz, who makes uſe of this notation in his anſwer to that letter. The rectangle contained by AQ and *oa* is equal to the rectangle contained by AP and *ox*; and hence rules for determining the fluxions of ſuch exponential quantities may be deduced by the third and tenth propoſitions: but we refer theſe to the ſecond book.

169. Suppoſe the logarithm AP to increaſe uniformly; let *pm* be the ſpace that would be deſcribed by the motion of *p* continued uniformly, in the time P deſcribes PG; and let *op*, *pm*, *mn*, *nr*, *rs*, &c. be a ſeries of terms in continued proportion. Then, if the fluxion of the logarithm AP be repreſented by PG, the firſt, ſecond, third, and all the higher fluxions of *op* ſhall be repreſented reſpectively by *pm*, *mn*, *nr*, and the ſubſe- Fig. 41.

quent terms of that geometrical progreſſion. For, ſince *pm* is to *op* in the invariable ratio of PG to *oa*, it follows from the fifth general theorem, that the fluxion of *pm* is meaſured by a line which is to *pm* (that meaſures the fluxion of *op*) as PG is to *oa*, or as *pm* is to *op*: but *mn* is ſuch a line, becauſe *mn* is to *pm* as *pm* is to *op*; and therefore *mn* meaſures the fluxion of *pm*, or the ſecond fluxion of *op*. In the ſame manner it appears, that *nr* meaſures the fluxion of *mn*, or the third fluxion of *op*; and that each higher fluxion is meaſured by a correſponding term of the progreſſion. It is evident, that when *op* is determined, and *pm* (which meaſures the firſt fluxion of *op*) varies, then *mn* (which meaſures the ſecond fluxion of *op*) varies in the duplicate ratio of *pm*; *nr* varies in the triplicate ratio of *pm*; and each term of the progreſſion varies in the ſame proportion as the power of *pm* that has its exponent equal to the number of terms which preceed that term in the ſeries.

170. We have ſhewn above, how to diſtinguiſh the increment of the fluent into ſuch parts as may be conceived to be generated in conſequence of its reſpective fluxions, and that bear a conſtant relation to their meaſures, in thoſe caſes when the fluent has no higher than firſt, ſecond and third fluxions. In order to illuſtrate what relates to the higher orders of fluxions, (which is repreſented as very abſtruſe,) we ſhall now chuſe an inſtance where fluxions of any order take place, and we never come to an invariable fluxion. Let *og*, *oa* and *op* be in continued proportion; and, *op* being greater than *oa*, (which we ſuppoſe invariable,) let the motion of *q* be uniform: then the velocity of *p* ſhall increaſe in the duplicate ratio of *op*, becauſe it is to the velocity of *q* as *op* is to *oq*, (art. 167.) or in the duplicate ratio of *op* to *oa*. While *q* deſcribes *kl*, let *p* deſcribe *ck*; and, *ob* being to *oa* as *oa* is to *oc*, and *ol* to *oa* as *oa* is to *ok*, it follows, that *ok* is to *oc* as *ob* is to *ol*, and *ck* to *ol* as *kl* is to *ol*. Let *cd* be to *dk*, *de* to *ek*, *ef* to *fk*, *fg* to *gk*, as *kl* is to *ol*; and the lines *cd*, *de*, *ef*, *fg*, *gu* ſhall be in a geometrical progreſſion that may be continued at pleaſure. The ſum of the terms of this progreſſion approaches continually to the increment *ck* as the ſeries is produced, ſo that their difference may become leſs than any aſſignable line. The line *ob* being

FIG. 42.

ſup-

ſuppoſed to flow uniformly, let its fluxion be repreſented by *hl*; and the firſt, ſecond, third, fourth, and all the higher fluxions of *oc*, ſhall be repreſented reſpectively by *cd*, 2*de*, 6*ef*, 24*fg*, and the ſubſequent terms of the progreſſion having coefficients prefixed to them that ariſe by the continual multiplication of the numbers 1, 2, 3, 4, 5, 6, &c. The line *ck* which is the increment of the fluent *oc* conſiſts of the parts *cd*, *de*, *ef*, *fg*, *gx*, &c. and we approach continually to its value by adding to *cd*, which repreſents the firſt fluxion of *oc*, one half of the line that repreſents its ſecond fluxion, one ſixth part of the line that repreſents its third fluxion, one twenty fourth part of the line that repreſents its fourth fluxion, and ſo on. The firſt part *cd* is what would have been generated, in the time *p* and *q* deſcribe *ck* and *hl*, if the motion of *p* had been continued uniformly from the term when it comes to *c*; the ſum of the two firſt parts *cd* and *de* (*viz. ce.*) is what would have been generated in the ſame time, if the acceleration of the motion of *p* had been continued uniformly from the ſame term, as in the 74th and 97th articles; the ſum of the firſt three parts *cd*, *de* and *ef* (*viz. cf*) is what would have been generated in the ſame time if the increaſe of that acceleration had been continued uniformly, as in the 130th and 134th articles, or the line that repreſents the ſecond fluxion of *op* had increaſed uniformly; and ſo on. To demonſtrate theſe things, ſuppoſe *oa*, *oc*, *om*, *on*, *or*, &c. to be in continued proportion; then *cd* ſhall be to *om* as *hl* is to *oa*, *de* to *on* in the duplicate ratio of *hl* to *oa*, *ef* to *or* in the triplicate ratio of *hl* to *oa*, and any term of the ſeries *cd*, *de*, *ef*, *fg*, *gx*, &c. is to the correſponding term of the ſeries *om*, *on*, *or*, *os*, &c. in a conſtant ratio, becauſe *hl* and *oa* are invariable. Therefore, by the fifth general theorem, the fluxion of *cd* is to the fluxion of *om* as *cd* is to *om*; and, in general, the fluxions of the correſponding terms in thoſe two progreſſions are in the ſame ratio as the terms themſelves. But, by art. 166. the fluxion of *om* is to the fluxion of *oc* as 2*om* is to *oc*, the fluxion of *on* is to the fluxion of *oc* as 3*on* is to *oc*, the fluxion of *or* is to the fluxion of *oc* as 4*or* is to *oc*; and ſo on. Therefore the fluxion of *cd* is to the fluxion of *oc* (which is repreſented by *cd*) as 2*cd* is to *oc*; and, 2*de* being to *cd* in the ſame

ratio of $2cd$ to oc, it follows, that the fluxion of cd is repreſented by $2de$. The fluxion of de is to the fluxion of oc as $3de$ is to oc, and therefore is repreſented by $3ef$; and thus it appears, that the fluxions of oc, cd, de, ef, fg, &c. are repreſented reſpectively by cd, $2de$, $3ef$, $4fg$, $5gx$, &c. The ſecond fluxion of oc is the ſame as the firſt fluxion of cd, and is therefore repreſented by $2de$. The third fluxion of oc is the ſame as the firſt fluxion of $2de$, and therefore is repreſented by $6ef$. The fourth fluxion of oc is the firſt fluxion of $6ef$, and is repreſented by $24fg$; and ſo on. While oq decreaſes, op increaſes; and, the fluxion of oq being conſidered as negative, all the fluxions of op are poſitive. When oq increaſes, op decreaſes; and, when p comes to c, all the fluxions of oc are repreſented by the ſame quantities as before; but they are alternately negative and poſitive when the fluxion of oq is invariable.

PROP. XII.

171. *Let op be greater than oa, ad to ap as oa is to op; and let oa, ad, de, ef, fg, &c. be in continued proportion: Then, by adding together ad, $\frac{1}{2}de$, $\frac{1}{3}ef$, $\frac{1}{4}fg$, &c. we approximate continually to the value of* AP *the logarithm of op.*

FIG. 43.

Let op, oa and oq be in continued proportion, and the motion of q from a towards o be uniform, as in the laſt article. Then, becauſe aq is to ap as oa is to op, ad is equal to aq, and the velocity of d is equal to the conſtant velocity of q. The velocity of P is to the velocity of p as oa is to op, (by art. 161.) and the velocity of p is to the velocity of q (or d) as op is to oq, (by art. 167.) Therefore the velocity of P is to the velocity of d as oa is to oq, or as op is to oa; ſo that the motion of P increaſes in the ſame ratio as op increaſes. The fluxion of AP is to the fluxion of ad in the ſame ratio of op to oa; and the terms oa, ad, de, ef, fg, &c. being in a continued geometrical progreſſion, the firſt term of which oa is invariable, it follows from the eighth propoſition, that the fluxion of ad is to the fluxions

xions of $\frac{1}{2}de$, $\frac{1}{3}ef$, $\frac{1}{4}fg$, &c. respectively, as oa is to ad, de, ef, &c. Therefore the fluxion of AP is to the fluxions of ad, $\frac{1}{2}de$, $\frac{1}{3}ef$, $\frac{1}{4}fg$, &c. respectively, as op is to the terms oa, ad, de, ef, &c. But, by adding together the terms oa, ad, de, ef, &c. we approximate continually to op; for ad is to dp, de to ep, ef to fp, &c. as oa is to ap; and the difference between op and the sum of the terms oa, ad, de, ef, &c. may become less than any assignable magnitude, by continuing the progression. Therefore, by adding together the fluxions of ad, $\frac{1}{2}de$, $\frac{1}{3}ef$, $\frac{1}{4}fg$, &c. we approximate continually to the value of the fluxion of AP; and, consequently, by summing up the lines ad, $\frac{1}{2}de$, $\frac{1}{3}ef$, $\frac{1}{4}fg$, &c. we approach continually to the value of AP the logarithm of op.

PROP. XIII.

172. *The same things being supposed as in the last proposition, we approximate continually to the logarithm of od by summing up the differences betwixt ad and $\frac{1}{2}de$, $\frac{1}{3}ef$ and $\frac{1}{4}fg$, $\frac{1}{5}gh$ and $\frac{1}{6}hk$,* &c.

Let od, oa and oz be in continued proportion, and oq shall be to oz as od is to op; so that qz shall be to dp as oq is to od. Let qz be divided in the same proportion in the points m, n, r, s, &c. as dp is divided in the points e, f, g, h, &c. and the sum of the terms oq, qm, mn, nr, rs, &c. shall approximate continually to oz. But as the difference of oa and ad (or aq) is equal to oq, so the difference of de and ef is equal to qn: for de is to ef as oa is to aq, (or ad,) by the supposition; and, consequently, df is to the difference betwixt de and ef as od is to oq, or as dp is to qz, and therefore as df is to qn. In the same manner, the difference of fg and gh is equal to ns; and, in general, the difference of any two terms which immediately succeed each other in the progression de, ef, fg, gh, &c. is equal to the sum of the two corresponding terms of the progression qm, mn, nr, rs, &c. Therefore, by summing up the differences betwixt oa and ad, de and ef, fg and gh, &c. we approximate continually to the value of oz. But the fluxion of AD, the logarithm of od, is to the fluxion

xion of *od* (or of *ad*) as *oa* is to *ad*, (by art. 161.) or as *oz* is to *oa*; and the fluxion of *ad* (by prop. 8.) is to the fluxions of the terms *ad*, $\frac{1}{2}de$, $\frac{1}{3}ef$, $\frac{1}{4}fg$, $\frac{1}{5}gb$, $\frac{1}{6}bk$, &c. respectively, as *oa* is to the terms *oa*, *ad*, *de*, *ef*, *fg*, *gb*, &c. Therefore we continually approximate to the value of the fluxion of AD by summing up the fluxions of the differences betwixt *od* and $\frac{1}{2}de$, $\frac{1}{3}ef$ and $\frac{1}{4}fg$, $\frac{1}{5}gb$ and $\frac{1}{6}bk$, &c. and, consequently, we approximate to the value of AD, the logarithm of *od*, by summing up these differences themselves.

173. Cor. Let *oa* be to *od* as *op* is to *ox*; and the logarithm of *ox* shall be equal to the sum of the logarithms of *op* and *od*; to which therefore we approximate by summing up $2ad$, $\frac{2}{3}ef$, $\frac{2}{5}gb$, &c. It is manifest, that *ox* is to *oa* as *od* is to *oq*; and that the logarithm of *ox* is that which measures the ratio of *od* to *oq*. But *od* and *oq* have half their sum equal to *oa* and half their difference equal to *ad*, which are the two first terms of the geometrical progression *oa*, *ad*, *de*, *ef*, *fg*, *gb*, &c. and it is obvious, that what has been shewn of these logarithms, coincides with what the excellent Dr. Halley demonstrates in the Philosophical Transactions, num. 216. This would lead us to the method of computing logarithms, but we refer that to the next book.

174. Hitherto we have supposed, with Lord Napier in his first scheme of logarithms, that, while *op* increases or decreases proportionally, the uniform motion of the point P, by which the logarithm of *op* (or the measure of the ratio of *op* to *oa*) is generated, is equal to the velocity of *p* at *a*, that is, at the term of time when the logarithms begin to be generated. But the uniform motion of P may be supposed equal to the motion of *p* at any other term, as when it comes to *e*; in which case the constant velocity of P is to the velocity with which *p* sets out from *a* when the logarithms begin to be generated, as *oe* is to *oa*; and thus we may have as many systems of logarithms as we please. The properties mentioned in the 156th and 157th articles, by which they become useful for facilitating computations, are common to all the systems. The line *oe* is what the learned Mr. Cotes calls the *Modulus* of the system. The measures of a given ratio in the different systems are in the same

propor-

proportion as the lines *oe*, *e* being always the term where the velocity of *p* becomes equal to the constant velocity of P. The logarithm of any quantity in NAPIER's first system becomes equal to the logarithm of the same quantity in any other system, whose modulus is *oe*, by multiplying it by the number which expresses the ratio of *oe* to *oa*; and the modulus of any system is to the modulus of any other system, as the logarithm of any given quantity in the first is to its logarithm in the second. Thus, in NAPIER's first scheme, in the same time that *op* from being equal to *oa* becomes equal to ten times *oa*, the point P describes a line the ratio of which to *oa* is that of 2.302 5809 &c. to unit. But it was found convenient, that the logarithm of ten should be supposed equal to unit; and the motion of P was supposed to be so far diminished, that the space described by it in that time might be equal only to *oa*; that is, its velocity in this case was supposed less than its velocity in the former in the same ratio as 1 is less than 2.302 5809 &c. If *oe* be taken less than *oa* in the same ratio, the velocity of P shall be equal in this case to the velocity of *p* at *a*; and *oe* shall be the modulus of this system, which therefore is expressed by 0.4342944 &c. *oa* being unit.

a *e* *o* *a* *ss*

p

A

P

175. When a ratio is given, its measure is always as the modulus of the system. It is therefore the same invariable ratio that is always measured by the modulus of the system, which is by Mr COTES called the *ratio modularis*. This ratio is that of *om* to *oe*, if *op* by increasing proportionally from being equal to *oe* become equal to *om*, in the same time that P by its uniform motion describes a line equal to the modulus *oe*.

176. While the base OP increases uniformly, let the ordinate *Pp* increase or decrease proportionally, (that is, let the velocity of *p* in the direction *Oo* be always as the ordinate *Pp*; and FIG. 44.

and

and the point *p* ſhall deſcribe the *logarithmic* curve. The baſe OP is always the logarithm of the ordinate Pp; and, if the uniform motion of P be equal to the motion of *p* at *o* in the direction *Oo*, then is the ordinate *Oo* the modulus of this ſyſtem of logarithms; and the fluxion of any ordinate is to the fluxion of the baſe as that ordinate is to *Oo*. The ordinates at equal diſtances from each other are always in geometrical progreſſion; from which it follows, that, when the baſe and curve are produced, they approach to each other continually on one ſide, ſo that their diſtance may become leſs than any aſſignable line; but, becauſe they can never meet, the baſe is therefore an aſymptote of the curve. When any two points of the curve, as *a* and *d*, are given, you may determine as many more points as you pleaſe, by raiſing ordinates upon the baſe at equal diſtances that are in any geometrical progreſſion of which the ordinates A*a* and D*d* are any two terms. When the curve is ſuppoſed to be deſcribed, the exponential quantities are eaſily determined by it. Let the ordinate A*a* be expreſſed by A; let OP be to OA as any quantity expreſſed by x is to unit: and the ordinate P*p* may be expreſſed by A^x, an exponential quantity of the firſt degree when x is variable.

FIG. 45. 177. Let *lmn* be an hyperbola, *o* its center, *or* and *os* its aſymptotes; let *al*, *pm* and *gn* parallel to the aſymptote *or* meet the curve in *l*, *m* and *n*. Let AL be equal to *al*, the angle LAP equal to *lop*, and AP be always the logarithm of *op* according to the 153d and 154th articles; then the hyperbolic area *almp* ſhall be always equal to the parallelogram LP. For, if *pg* be to PG as the velocity of *p* is to the velocity of P, that is (by art. 161.) as *op* is to *oa*, or as *al* is to *pm*, (by the property of the hyperbola;) then ſhall *pg* be to PG as PM is to *pm*, the parallelogram *mg* ſhall be always equal to the parallelogram MG, and (by prop. 4.) the fluxion of the hyperbolic area *almp* equal to the fluxion of the parallelogram LP. Therefore, by the fourth general theorem, the hyperbolic area *almp* is equal to the parallelogram LP, and the area *almp*, or the ſector *oml*, (which is equal to it becauſe the triangles *opm*, *oal* are equal) meaſures the ratio of *op* to *oa*. Theſe areas form a ſyſtem of logarithms the modulus of which is the parallelogram *ol*, and

ſerve

ſerve for meaſuring ratios, as the arches or ſectors of circles ſerve for meaſuring angles. The former are divided into equal parts, by reſolving a ratio into the equal ratios from which it is compounded; as, by dividing the latter into equal Parts, an angle is reſolved into the equal angles of which it conſiſts.

178. The area of the parabola admits of a perfect quadrature, and elliptic areas are reduced to ſuch as are circular; therefore the areas of all the conic ſections may be reduced to rectilineal figures, the meaſures of angles, or the meaſures of ratios: and to one or more of theſe all fluents ought to be reduced in this doctrine as much as poſſible. As a conic ſection paſſes from one ſpecies to another, by varying the inclination of the plane that cuts the cone; or by varying the circumſtances of the deſcription, when the curve is traced upon a plane by motion; or by changing a ſign, coefficient or exponent of a term, when the nature of the curve is expreſſed by an equation: ſo the expreſſions of the meaſures of angles and of ratios are by ſimilar variations tranſformed into each other, and in ſome caſes into ſuch as repreſent rectilineal figures; and, by a change in the ſign, coefficient or exponent of a term in the expreſſion of a fluxion, the nature of the fluent is ſo far altered that it becomes reducible to a conic ſection of a different kind. When *op* increaſes or decreaſes proportionally, the acceleration or retardation of the motion of *p* is as its diſtance from the given point *o*; but there are alſo other caſes in which its acceleration or retardation obſerves the ſame law: and, by taking theſe together, we may comprehend the geneſis of the lines that meaſure ratios and angles in one view. But we ſhall have occaſion afterwards to conſider further the analogy there is betwixt thoſe meaſures, and to treat of the logarithms that are called *imaginary*.

179. In NAPIER's firſt ſyſtem, the conſtant velocity of P, FIG. 46. by which the logarithm AP is generated, is equal to the velocity with which *p* ſets out from *a* when their geneſis is ſuppoſed to begin, (art. 153. & 154.) Therefore, if we ſuppoſe AP and *ap* to be diminiſhed continually, (AP being always the logarithm of *op*,) their ratio ſhall approach continually to a ratio of equality as its limit, (by art. 67.) and, *oa* being ſuppoſed to repreſent unit, if *ap* be very ſmall compared with *oa*, it may be ſuppo-

ſuppoſed in approximations to be the logarithm of *op* in this ſyſtem, which is called its *hyperbolic logarithm*; and any very ſmall fraction may be ſuppoſed to be the logarithm of the ſum of unit and that fraction added together. From this it follows, that, if a ſeries of mean proportionals be interpoſed betwixt *oa* and any given line *ob*, and the number of all the terms without including *oa* be *n*, and *on* being the ſecond term of the ſeries *an* be to *ab* as unit is to the number *q*; then, by increaſing continually the number of mean proportionals betwixt *oa* and *ob*, the ratio of *x* to *q* ſhall approach continually to the ratio of AB (the logarithm of *ob*) to *ab*, as its limit. For the number *x* is to unit as AB is to AN, (the logarithm of *on*;) and unit is to *q* as *an* (which approaches continually to AN) is to *ab*. Therefore the ratio of *x* to *q* approaches continually to that of AB to *ab*. For example, if *ob* be double of *oa*, the ratio of *x* to *q* * approaches continually to the ratio of the logarithm of 2 to unit, which is nearly that of 7 to 10.

CHAP. VII.

Of the Tangents of curve Lines.

180. AN arch of a curve has its concavity turned one way, when the right lines that join any two of its points are all on the ſame ſide of the arch; or, in general, (that we may include, with ARCHIMEDES †, ſuch lines as have rectilineal parts,) a line has its concavity turned one way, when the right lines that join any two of its points are either all upon one ſide of it, or while ſome fall upon the line itſelf, none fall on the oppoſite ſide.

* Thus, if the fraction $1 + \frac{1}{q}$ (repreſented here by *on*) raiſed to the power *x* be ſuppoſed equal to 2, the ratio of *x* to *q* ſhall be nearly that of the logarithm of 2 to unit, when *q* is a large number. See the *Doctrine of Chances, probl.* 3. by the excellent Analyſt Mr. de Moivre.

† De ſphæra & cylindro, defin. 2.

Fig. 39.

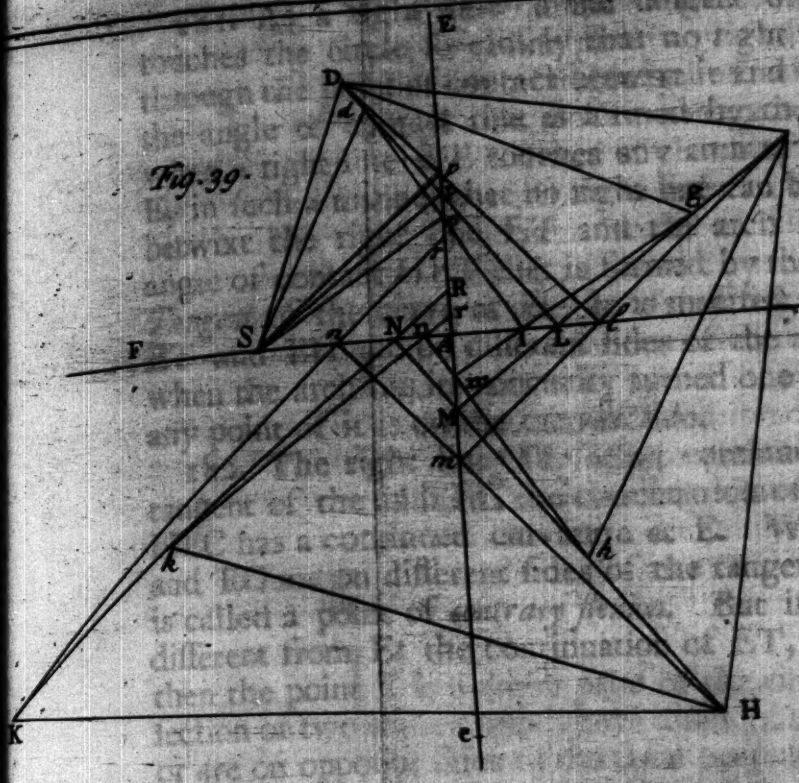


Fig. 40.

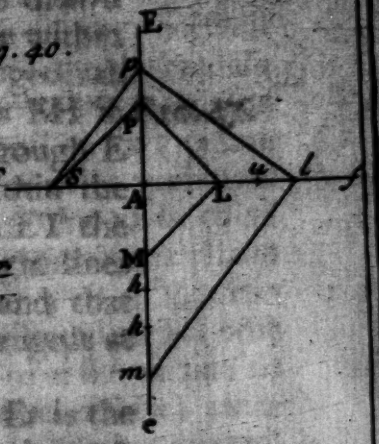


Fig. 45

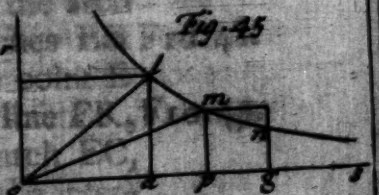


Fig. 41

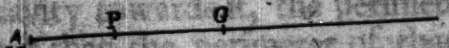


Fig. 42

Fig. 43

Fig. 46

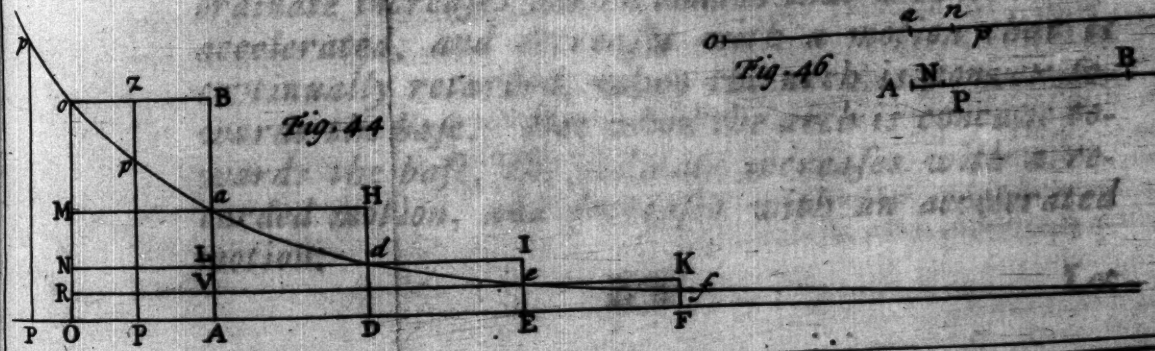
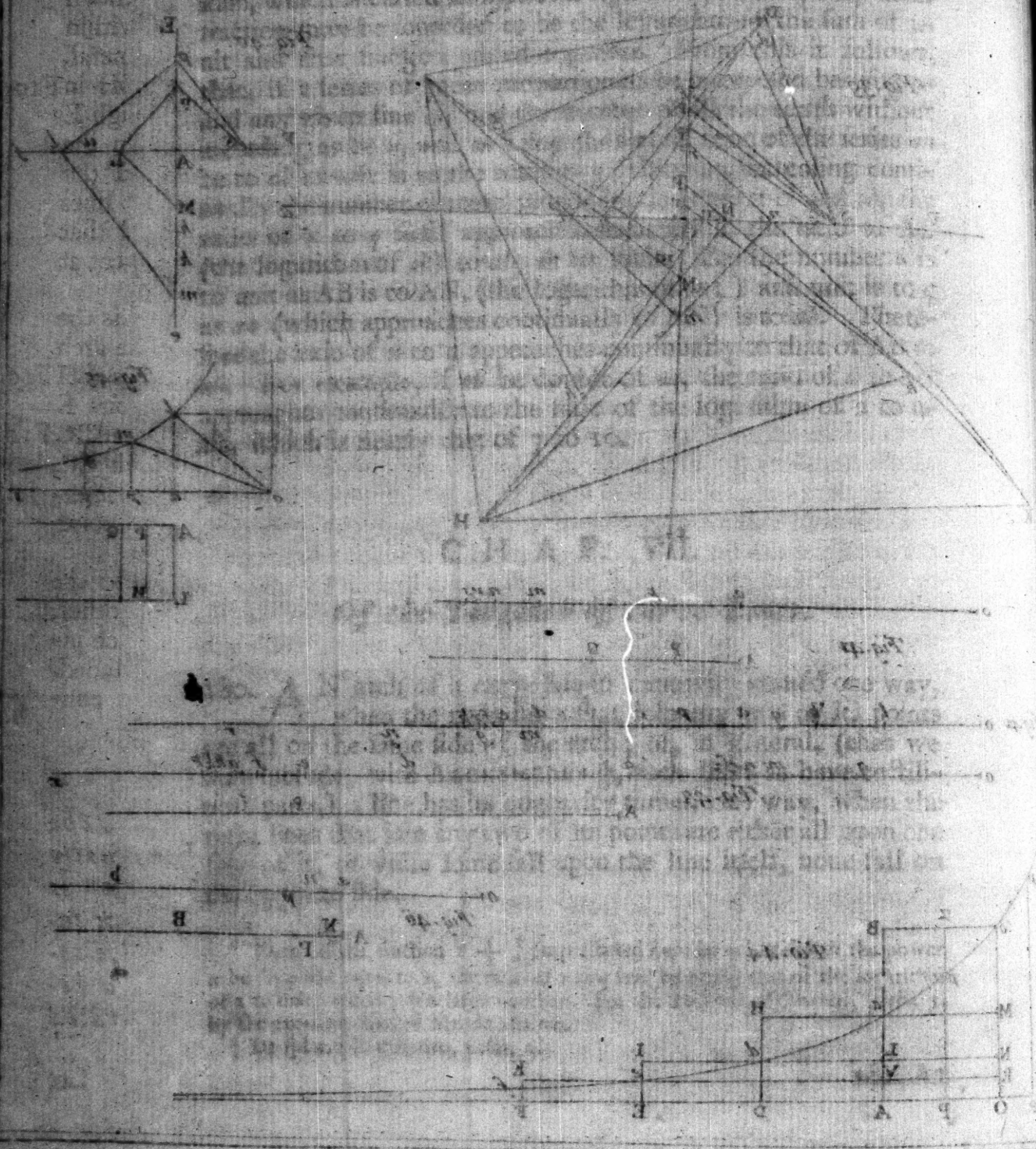


Fig. 44



181. As a right line is the tangent of a circle, when it touches the circle so closely that no right line can be drawn through the point of contact betwixt it and the arch, or within the angle of contact that is formed by them; so, in general, when a right line ET touches any arch of a curve, as EH in E, in such a manner that no right line can be drawn through E betwixt the right line ET and the arch EH, or within the angle of contact HET that is formed by them, then is ET the *Tangent* of the curve at E. It is manifest, that the right lines ET and EH are on different sides of the arch EH; and that when the arch has its concavity turned one way, the tangent at any point of it is on the convex side. FIG. 47.

182. The right line TE being continued to *t*, if E*t* is the tangent of the arch EC the continuation of HE, then the arch HEC has a continued curvature at E. When the arches EH and EC are on different sides of the tangent TE*t*, the point E is called a point of *contrary flexion*. But if any right line ER, different from E*t* the continuation of ET, touch the arch EC, then the point E is a *double point* of the curve, and is the intersection of two arches which have different tangents at that point, or are on opposite sides of the same tangent. FIG. 48. FIG. 49.

183. When two lines that have their concavity turned the same way have the same terms, and the one includes the other, or has its concavity towards it, the perimeter of that which includes is greater than the perimeter of that which is included. This is the second axiom of the treatise of ARCHIMEDES concerning the sphere and cylinder.

LEMMA VII.

184. *The base being supposed to flow uniformly, the ordinate increases with a motion that is continually accelerated, and decreases with a motion that is continually retarded, when the arch is convex towards the base. But when the arch is concave towards the base, the ordinate increases with a retarded motion, and decreases with an accelerated motion.*

Fig. 47. Let the arch be convex towards the base, and let the ordinate Pp increase while the base AP increases. Let BD and DG be equal lines described by P with an uniform motion in any equal times that immediately succeed after each other. Let the ordinates BC, DE and GH meet the curve in the points C, E and H; let the the tangent at E meet the ordinates GH and BC in T and t; and let MEI parallel to the base meet these ordinates in I and M. Then, because BD and DG are equal, IT is equal to Mt; and, the arch CEH being convex towards the base, IH is greater than IT, or Mt; which is itself greater than MC, for the same reason. Therefore IH is surely greater than MC. But MC and IH are the increments of the ordinate that are generated in equal times while the base acquires the augments BD and DG; and, since those increments continually increase, it follows, that the motion with which the ordinate flows is continually accelerated. When the ordinate decreases, the decrements IH, MC continually decrease, and the ordinate flows with a motion that is continually retarded.

Fig. 50. 185. Let the arch be concave towards the base; and, the construction being the same, MC shall exceed Mt, or IT, which is itself greater than IH: so that MC being greater than IH, the ordinate flows with a motion that is retarded or accelerated continually according as it increases or decreases while the base increases.

Fig. 47. 186. The motion of the point p, which is supposed to describe the curve, is perpetually accelerated, when, the base being supposed to flow uniformly, the curve is convex towards the base, and the ordinates increase while the curve increases. For the arch EH being greater than its chord EH, which in this case exceeds the tangent ET, (because the point T is betwixt I and H, and the angle ETH is greater than EHT,) the arch EH is surely greater than the tangent ET. But the arch EC is less than the tangent Et, or ET; for if CV parallel to the base meet the tangent Et in V, VC shall be less than Vt; and, the arch EC being less than the sum of EV and VC, (by art. 183.) it must be less than Et. Therefore the arch EH is greater than CE, and the motion of p in describing the curve is perpetually accelerated. In the same manner it appears, that,

when

when the curve is convex towards the baſe, and the ordinates decreaſe while the curve increaſes, the motion of p (or that with which the curve Fp flows) is retarded perpetually.

187. But when the curve is concave towards the baſe, and the baſe being ſuppoſed to flow uniformly, the curve increaſes while the ordinates increaſe, the arch EC is greater than the chord EC, and therefore greater than the tangent Et, or ET, which is itſelf greater than the ſum of EV and VH, (VH parallel to the baſe being ſuppoſed to meet the tangent in V,) and conſequently is greater than the arch EH, (by art. 183.) Therefore, in this caſe, the lines deſcribed by p in equal ſucceeding times perpetually decreaſe, and its motion (or the motion with which the curve Fp flows) is perpetually retarded. But if the ordinates decreaſe, in this caſe, while the curve increaſes, the motion of p is perpetually accelerated. FIG. 50.

PROP. XIV.

188. *Let* ET *be the tangent of the curve* FE *at* E; *and,* EI *being parallel to the baſe* AD, *let* IT *be parallel to the ordinate* DE: *Then the fluxions of the baſe, ordinate and curve, ſhall be meaſured by the right lines* EI, IT *and* ET, *reſpectively.*

Firſt, let the arch CEH be convex towards the baſe; and the baſe being ſuppoſed to flow uniformly, the ordinate ſhall increaſe with an accelerated motion, by the laſt lemma. The figure and conſtruction being the ſame as in the 184th article, it follows from the firſt and ſecond axioms, that, in the ſame time that the baſe acquires the augment DG, a line leſs than IH, but greater than MC, would be generated by the motion with which the ordinate DE flows continued uniformly. Therefore, if this line be not equal to IT, firſt let it be equal to ſome line IL, leſs than IH, but greater than IT. Join EL, and it ſhall meet the curve EH in ſome point R betwixt E and H; let the ordinate RQ meet the tangent in N, the baſe in Q, and the right line EI in K. Then, ſince IL is to KR as EI is FIG. 46.

is to ER, or as DG is to DQ, and, when the generating motion is uniform, the quantities generated are in the ſame proportion as the times; it follows, that, if the motion with which the ordinate DE flows was continued uniformly, the line KR would be generated by it in the ſame time that the baſe acquires the augment DQ. But, becauſe the ſame line KR is generated by the accelerated motion with which the ordinate flows, in the ſame time that the baſe by flowing uniformly acquires the augment DQ, it follows from the firſt axiom, that a leſs line than KR would be generated in that time by the motion with which the ordinate DE flows continued uniformly. And theſe being contradictory, it follows, that the line that would be generated by the motion with which the ordinate DE flows continued uniformly while the baſe acquires the augment DG, is not greater than IT. If it be ſaid to be leſs than IT, or M*t*, let it be equal to M*l*; and, M*l* being greaterthan MC, but leſs than M*t*, it follows, that E*l* muſt meet the arch CE in ſome point *r* betwixt E and C. Let the ordinate *rq* meet the tangent in *n*, the baſe in *q*, and the right line EM in *k*. Then, ſince M*l* is to *kr* as EM is to E*k*, or as DB is to D*q*; it follows, that, if the motion with which the ordinate DE flows was continued uniformly, it would generate a line equal to *kr* in the time that the baſe acquires an augment equal to D*q*. But, ſince the increment *kr* is generated in an equal time by the accelerated motion with which the ordinate flows before *p* comes to E, that is, before the motion with which the ordinate DE flows is acquired; it follows from the ſecond axiom, that a line greater than *kr* would be generated in the ſame time by the motion with which DE flows continued uniformly. And theſe being alſo contradictory, it follows, that IT meaſures accurately the motion with which the ordinate DE flows, or its fluxion, when the motion with which the baſe flows, or its fluxion, is meaſured by DG or EI.

189. In the ſame caſe, the motion of the point *p*, that is ſuppoſed to deſcribe the curve, is perpetually accelerated by art. 186. If the motion of *p* was continued uniformly from E, a line leſs than the arch EH, but greater than the arch CE, would be deſcribed by it in the time P deſcribes DG, by the firſt and

ſecond

second axioms. If this line be not equal to the tangent ET, first let it exceed ET by TL; and, because this line is less than the arch EH which is less than the sum of ET and TH, (by art. 183.) it follows, that TL is less than TH, and that the right light line EL must meet the arch EH in some point R betwixt E and H. Let the ordinate RQ meet the base in Q and the tangent in N. Then, since ET is to EN, and TL to NR, as DG is to DQ; it follows, that the sum of ET and TL is to the sum of EN and NR as DG is to DQ; and that a line equal to the sum of EN and NR would be described by *p*, if its motion was continued uniformly from the term when it comes to E, in the time P describes DQ. But, while the motion of P is uniform, the motion of *p*, in describing the arch ER, is perpetually accelerated, (by art. 186.) and it follows from the first axiom, that, in the time P describes DQ, a line less than the arch ER (and consequently less than the sum of EN and NR) would be described by *p* if its motion was continued uniformly from E. And these being contradictory, it follows, that the line which would be described by *p* if its motion was continued uniformly from E, in the time P describes DG, is not greater than ET. If this line be said to be less than ET, or E*t*, let E*t* exceed it by *tl*; and, since that line is greater than the arch EC (which is greater than the chord EC, and therefore is greater than the excess of E*t* above *t*C,) it follows, that *tl* must be less than *t*C, and that the right line E*l* must meet the arch EC in some point *r* between E and C. Let the ordinate *rq* meet the base in *q*, and the tangent in *n*: and, since E*t* is to E*n*, and *tl* is to *nr*, as DB is to D*q*; it follows, that, if the motion of *p* was continued uniformly from E, a line equal to the difference of E*n* and *nr* would be described by it in the time P describes a line equal to *q*D. But, the motion of *p* being perpetually accelerated, the same line must be greater than the arch *r*E, (by ax. 2.) and therefore greater than the difference of E*n* and *nr*, which is less than E*r*, by art. 183. And these being also contradictory, it appears, that the motion of *p* at the term when it comes to E, or the fluxion of the curve FE, is measured accurately by ET when the fluxion of the base is measured by DG, or EL.

190. When

Fig. 50. 190. When the curve is concave towards the base, the proposition is demonstrated in the same manner from the third and fourth axioms: or it may be deduced from the preceeding case, by drawing any right line *ao* parallel to the base in such a manner that the arch CEH may have its convexity towards it. For, if DE produced meet *ao* in *d*, and the curve meet *ao* in *f*, the motions with which *d*E and *f*E decrease are equal to those with which DE and FE increase. When the base flows with a variable motion, the proposition is demonstrated from what we have shewn by the eleventh general theorem, and the 60th article.

Fig. 47. 191. Cor. In the first case, when the curve is convex towards the base, the part TH of the increment IH of the ordinate DE that is generated while the base acquires the augment DG, is owing to the acceleration of the motion with which the ordinate flows during that time. When the curve is concave
Fig. 50. towards the base, the increment IH of the ordinate is less than IT; and the difference arises from the retardation of the motion with which the ordinate flows. The fluxion of the ordinate is the same in all curves that have the same tangent at E.

Fig. 51. 192. The point S, the right line AO, and the circle FDB described about the center S, being given in position, let the point P describe the right line AO; SP meet the circle FDB in N; and let the point M describe any right line SD so that SM may be always equal to SP. Let SA be perpendicular on AO; and, if the motion of the point P from A towards O be uniform, the motion of the point M in the right line SD shall be perpetually accelerated; but the motion of the point N shall be perpetually retarded. For let *b*E, EH be equal lines described by P in any equal times. Let S*b*, SE and SH meet the circle FDB in *g*, D and G. From S as center, through *b* and H, describe the arches *bk*, HK meeting SE in *k* and K, and let *bl*, HL be perpendicular on SE in *l* and L. Then shall *k*E and EK be the lines described by M, and *g*D, DG the arches described by N in the same equal times. Because *b*E is equal to EH, therefore (Elem. 26. 1.) E*l* is equal to EL; and, EK being greater than EL, or E*l*, which is greater than E*k*, it is evident that EK is greater than E*k*; and therefore the motion of M is pepetually accelerated. Because HL is equal to *bl*, but

SH

SH greater than S*b*, the angle HSL is leſs than *b*S*l*, and the motion of N is perpetually retarded. It is obvious, that, when the motion of M from S towards N is uniform, the motions of P and N are retarded; and that, when the motion of N from *f* towards *b* is uniform, the motions of P and M are perpetually accelerated.

PROP. XV.

193. *The point* S *and the right line* AO *being given in poſition, let the circle* f N*b deſcribed from the center* S *meet* AO *in* E; *and the fluxions of the right lines* AE, SE *and of the arch* f E *ſhall be to each other in the ſame ratio as the right lines* SE, AE *and* SA. FIG. 52.

While the point P deſcribes AO, let SP meet the circle f E*b* in N, and upon SE let SM be always equal to SP, as in the preceeding article. Then the fluxions of the right lines AE, SE and of the arch f E ſhall be meaſured by the velocities of the points P, M and N at the term when they come together to E. Let the motion of P be uniform, and the motion of M ſhall be accelerated, by the laſt article. It is manifeſt, that the velocity of P is greater than the velocity of M. If the velocity of P be to the velocity of M, at the term when they come to E, in a leſs ratio than that of SE to AE, let their ratio be the ſame as that of SH to AH, or (EK being perpendicular on SH in K) that of EH to KH; and the point M ſhall deſcribe a line equal to KH, if its motion be continued uniformly from E, in the time P deſcribes EH; and, KH being greater than GH the exceſs of SH above SE, it follows, that a greater line ſhall be deſcribed in the ſame time by M, when its motion is continued uniformly from E, than when it is continually accelerated from the ſame term, againſt the firſt axiom. If the velocity of P be to the velocity of M at the term when they come to E in a greater ratio than that of SE to AE, let their ratio be the ſame as that of SC to AC, or (E*k* being perpendicular on SC*g* in *k*)

that of EC to Ck; and the point M ſhall deſcribe a line equal to Ck by its motion continued uniformly from E in the ſame time P deſcribes a line equal to CE; and, Ck being leſs than Cg the exceſs of SE above SC, it follows, that the line deſcribed by an accelerated motion is greater than the line which is deſcribed in an equal time by the motion acquired by this acceleration continued uniformly, againſt the ſecond axiom. Therefore the velocity of P is to the velocity of M, at the term when they come to E, as SE is to AE. This might have been demonſtrated from the 96th article in a different manner, becauſe the fluxion of the ſquare of SE is equal to the fluxion of the ſquare of AE.

194. The motion of P from A towards O being uniform, the motion of N is perpetually retarded, by the 192d article. If the velocity of P be to the velocity of N as SH greater than SE is to SA, or as EH is to EK, the point N ſhall deſcribe EK, by its motion continued uniformly from E, in the time P deſcribes EH; and, EK being leſs than the arch EG which is deſcribed in the ſame time by N with a retarded motion, it follows, that a leſs line would be deſcribed in the ſame time by a motion continued uniformly, than by the ſame motion perpetually retarded from the ſame term, againſt the third axiom. If the velocity of P be to the velocity of N in a leſs ratio than that of SE to SA, or (CR being perpendicular in R on ET the tangent of the circle at E) that of CE to ER, let it be the ſame as that of CE to Ex; join Cx, and let Sz parallel to it meet AE in Q, the arch Eg in *u*, and the tangent ET in *z*. Then, ſince Ex is to Ez as EC is to EQ, it follows, that the point N would deſcribe a line equal to Ez, by its motion continued uniformly from E, in the time P deſcribes a line equal to EQ. But the point N deſcribes the arch *u*E by its retarded motion in that time before it comes to E; and, the arch *u*E being leſs than its tangent Ez, it follows, that the line deſcribed by N while its motion is retarded, is leſs than the line which would be deſcribed in an equal time by the motion that remains after that retardation continued uniformly, againſt the fourth axiom. Therefore the velocities of the points P, M and N, at the term when they come together to E, are in the ſame ratio as the right lines

lines SE, AE and SA; and the fluxions of the right lines AE, SE and of the arch *f*E are in the same ratio. This may be shewn in like manner, by supposing the motion of M or that of N to be uniform; and the proposition is made general by the eleventh theorem.

195. Cor. I. Let a circle A*n*D described from the center S meet SP in *n*, and SE in D; and the velocity of *n* shall be always to the velocity of N as S*n* or SA is to SN or SE, by theor. 3. The velocity of N is to the velocity of P when they come to E in the same ratio. Therefore the velocity of *n* at D is to the velocity of P at E in the duplicate ratio of SA to SE; and the fluxion of the arch AD is to the fluxion of its tangent AE in the same ratio. In the same manner it appears, that the fluxion of the arch AD is to the fluxion of its secant SE as the square of SA is to the rectangle SEA.

196. Cor. II. If the points P and *p* set out with equal velocities from A and *a* in the right lines AO, *ao* at the same time; and, the motion of *p* being continued uniform, the velocity of P be always as its distance from S; then *ap* shall be the logarithm of the sum of SP and AP. For the fluxion of *ap* shall be to the fluxion of AP as SA is to SP, and to the fluxion of SP as SA is to AP. Therefore the fluxion of *ap* shall be to the sum of the fluxions of SP and AP as SA is to the sum of SP and AP; and, when *ap* increases uniformly, the sum of SP and AP increases proportionally, (by art. 160.) Therefore *ap* is the logarithm of the sum of SP and AP, or is the measure of the ratio of that sum to SA, the modulus being SA. The velocity of P is to the velocity of *p* as the velocity of *p* is to the velocity of *n*; and, when P comes to E, the velocity of *p* is equal to the velocity of N.

197. Cor. III. Let the right line *ao* given in position meet AO in F, and SP always intersect *ao* in *p*; then the velocity of P shall be to the velocity of *p* as the rectangle SPF is to the rectangle S*p*F; and, if S*a* be perpendicular on *ao*, the fluxion of AP shall be to the fluxion of *ap* in the same ratio. This may be deduced from the 15th proposition, or immediately from the axioms, thus: If *ao* meet SA in some point betwixt S and A, the motion of *p* shall be accelerated when the motion of Fig. 53.

of P from A towards O is uniform. For, while P deſcribes any equal lines CE, EH, let *p* deſcribe *ce* and *eb*; let *le*L parallel to AF meet SH and SC in L and *l*; and it is manifeſt, that L*e* ſhall be equal to *el*, but that *eb* ſhall be greater than *ec*; and therefore the motion of *p* is accelerated. Let L*k* parallel to SE meet *eb* in *k*; and, if the motion of *p* was to be continued uniformly from *e*, the line *ek* would be deſcribed by it in the ſame time P deſcribes EH. For, if it ſhould be ſaid, that it would deſcribe *er* greater than *ek* in that time, let S*q* parallel to L*r* meet EH in Q, *e*L in *x*, and *eb* in *q*. Then, ſince *er* is to *eq* as *e*L is to *ex*, or as EH is to EQ, it follows, that the point *p*, by its motion continued uniformly from *e*, would deſcribe *eq* in the time P deſcribes EQ. But the point *p* deſcribes the ſame line *eq* in the ſame time when its motion is continually accelerated from that term; ſo that the ſame ſpace would be deſcribed in the ſame time by *p* when its motion is continued uniformly from *e*, and when it is continually accelerated, againſt the firſt axiom. In like manner, it is ſhewn from the ſecond axiom, that the line which would be deſcribed by the motion of *p* continued uniformly from *e*, in the time P deſcribes EH, is not leſs than *ek*. Therefore that line is equal to *ek*; and the velocity of *p* is to the velocity of P, when they come to *e* and E, as *ek* is to EH; and the fluxion of *ae* is to the fluxion of AE in the ſame ratio. But the ratio of *ek* to EH is compounded of the ratio of *ek* to *e*L, or that of *e*F to EF, and of the ratio of *e*L to EH, or that of S*e* to SE. Therefore the fluxion of *ae* is to the fluxion of AE as the rectangle S*e*F is to the rectangle SEF. When the motion of *p* is retarded while that of P is uniform, the demonſtration is deduced in the ſame manner from the third and fourth axioms: or the ſame demonſtration may ſerve, by ſuppoſing the motion of *p* in thoſe caſes uniform; for the motion of P ſhall then be accelerated. When the points E and *e* come to F, *ek* is to EH as SA is to S*a*; and the fluxion of *a*F is to the fluxion of AF in that ratio. When *ao* is parallel to AO, it is manifeſt, that the fluxion of *ae* is to the fluxion of AE as S*e* is to SE, which in this caſe is an invariable ratio, and is the ſame as that of S*a* to SA.

198. Cor. IV. The fluxion of SE is to the fluxion of AE as

as AE is to SE; and the fluxion of *ae* is to the fluxion of S*e* as S*e* is to *ae*. Therefore the fluxion of SE is to the fluxion of S*e* as the rectangle AEF is to the rectangle *ae*F. Let S*u* constitute the angle *e*S*u* equal to ASF, and meet *e*F in *u*; then the fluxion of SE shall be to the fluxion of S*e*, when the points E and *e* come to F, as the rectangle contained by AF and S*e* is to the rectangle contained by *e*F and SA, or as *eu* is to *e*F.

199. ARCHIMEDES demonstrates, in the 6th propoſition of his treatiſe concerning ſpiral lines, that a right line SP may be drawn from S meeting AE in P, and the circle *f*E in N, ſo that PN may be to the chord EN in any ratio leſs than that of AE to SA. For, if SI parallel to AE meet the chord NE and tangent ZE produced in V and I, PN ſhall be to EN as SN or SE is to NV; and, ſince NV may be equal to any right line greater than EI, it follows, that PN may be to EN in any ratio leſs than that of SE to EI, or that of AE to SA. It is alſo manifeſt, that as N approaches to E, NV decreaſes, and that the ratio of SE to NV or of PN to EN continually decreaſes. In like manner he ſhews, in the 7th propoſition, that a right line S*p* may be drawn from S meeting the right line AE produced beyond E in *p*, and the circle in *n*, ſo that *pn* may be to the chord E*n* in any ratio greater than that of AE to SA, or that of SE to EI; becauſe, if E*n* produced meet SI in *v*, *pn* ſhall be to E*n* as S*n* or SE is to *nv*, and *nv* may be equal to any line leſs than EI. In this caſe, when *n* approaches to E, *nv* increaſes, and the ratio of *pn* to E*n* or of SE to *nv* continually decreaſes. FIG. 54.

200. In the 8th propoſition of the ſame treatiſe, he ſhews that PN may be to the tangent EZ in any ratio leſs than that of AE to SA, as that of SE to EL, EL being greater than EI. For let a circle deſcribed through L, S and I meet SE produced in R: then, becauſe EL is greater than EI, a right line SZY may be drawn from S meeting this circle in Y, ſo that ZY may be equal to ER. Suppoſing that SZ is ſuch a right line, the rectangle SZY being equal to LZI, and the rectangle contained by SZ and EI being equal to the rectangle contained by SP and ZI, becauſe SZ is to SP as ZI is to EI, it follows, that the rectangle LZI is to the rectangle contained by SP and ZI as the FIG. 55.

rectangle

rectangle SZY is to the rectangle contained by SZ and EI. Therefore LZ is to SP = ZY or ER is to EI, or as EL is to ES or NS; and EZ is to PN as EL is to SE. It is easy to see, that when N approaches to E, the ratio of PN to the tangent EZ increases continually. In like manner he shews, in the 9th proposition, that *pn* may be to the tangent Ez in any ratio that is greater than that of SE to EI, or of AE to SA; and it is easy to shew, that, when *n* approaches to E, the ratio of *pn* to Ez decreases continually.

Fig. 56. 201. Let P describe an arch FPH of a continued curvature; let SP meet the circle *f*E*b* in N; and suppose that SM is always taken equal to SP, as in art. 192. Then, if SP increase continually, and, the arch being supposed to have its convexity towards S, if the motion of N be uniform, the motion of M shall be perpetually accelerated. For, if the arch EN be equal to E*n*, and the right lines SN, S*n* meet the curve in P and *p*, and its tangent at E in T and *t*, E*t* shall be greater than ET, and the excess of S*p* above SE being greater than the excess of S*t* above SE, which is greater than the excess of SE above ST, it must be greater than the excess of SE above SP. In this case, the motion of P in the curve is also accelerated; for it is manifest, that the arch E*p* is greater than the arch EP. When the arch FPH is concave towards S, and the motion of N is uniform, the motion of M may be uniform, accelerated or retarded.

PROP. XVI.

202. *Let the circle f*E*b meet the curve* FPH *in* E, *and* SD *perpendicular on* SE *meet the tangent* ET
Fig. 57. *in* D; *then the fluxion of the ray* SE *shall be to the fluxion of the arch f*E *as* SE *is to* SD.

Case 1. The motion of N in the arch *f*E*b* being uniform, let the motion of M in the line SE be also uniform; in which case P describes the spiral of Archimedes. The fluxion of SM is to the fluxion of *f*N as the constant velocity of M is to the constant velocity of N, or as EM is to the arch EN. If the ratio

Pl. VII. Pa. 190.

Fig. 50

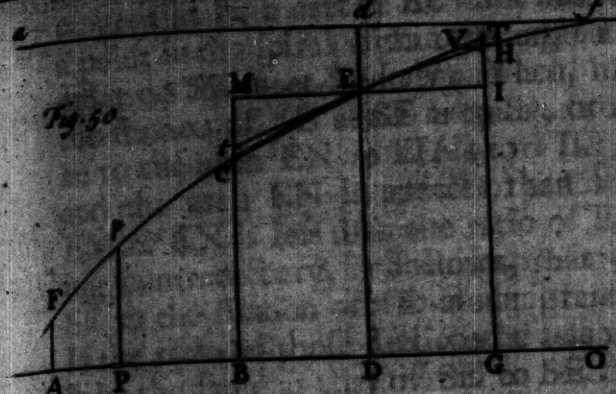
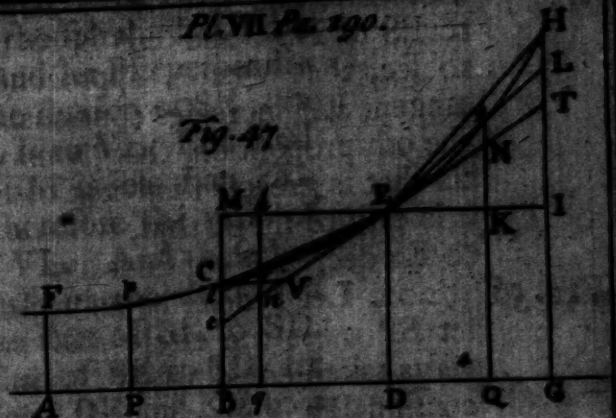


Fig. 47



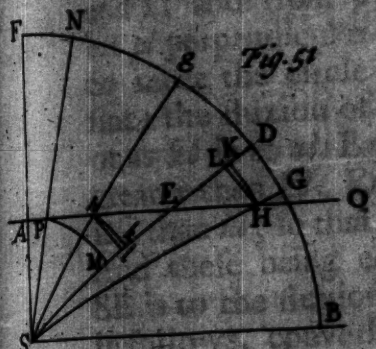


Fig. 51

Fig. 52

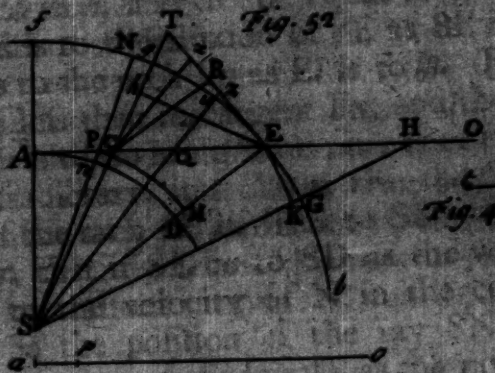
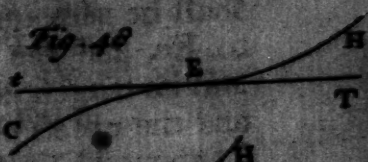


Fig. 48



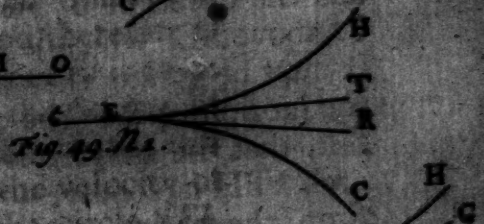


Fig. 49 N. 1.

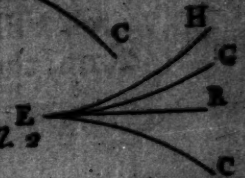


Fig. 49 N. 2

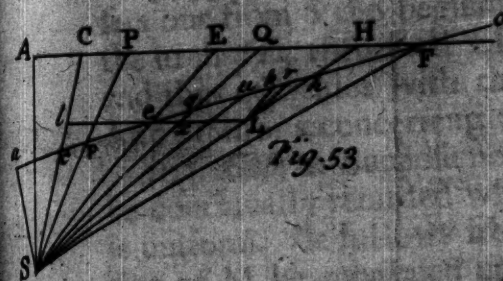


Fig. 53

Fig. 55

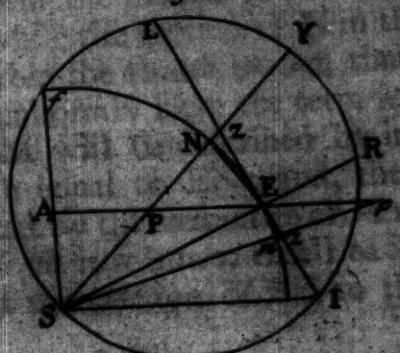


Fig. 56.

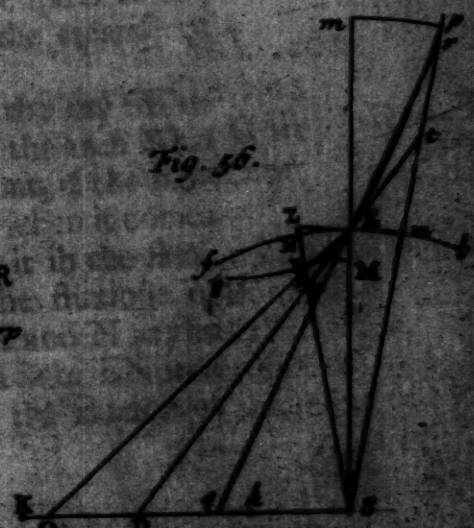
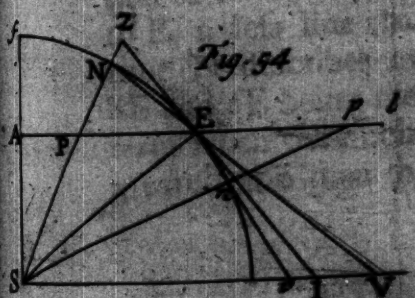
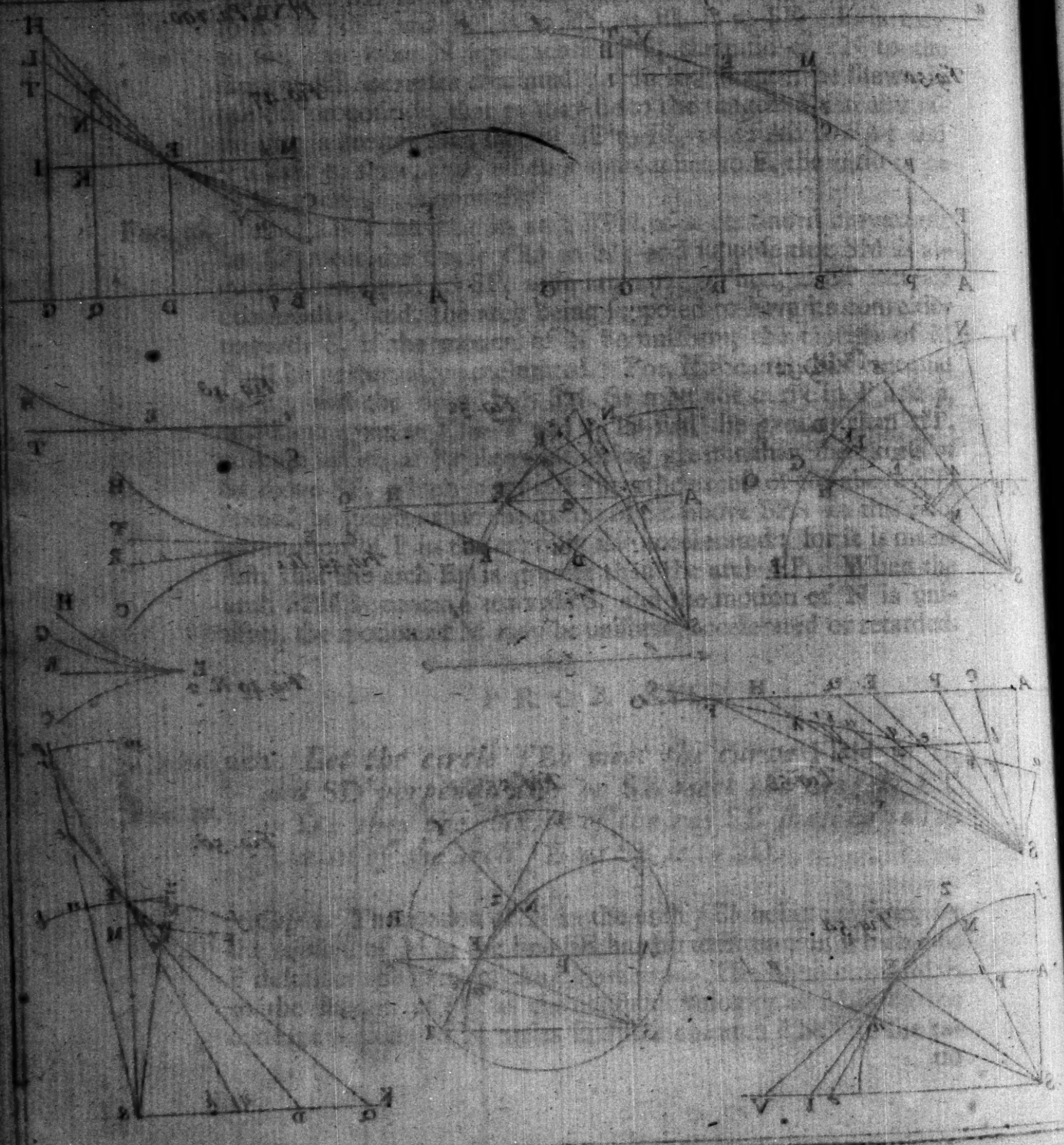


Fig. 54





tio of thoſe fluxions be greater than that of SE to SD, let it be the ſame as that of SE to S*k*; join E*k*, and, ſince ED is the tangent at E, a part of E*k* muſt fall within the ſpiral. Let P be a point in the ſpiral within the angle *k*ED, and let PL perpendicular on SE meet E*k* in V. Then, ſince the fluxion of SE is to the fluxion of *f*E as SE is to S*k*, or as EL is to VL, EM ſhall be to the arch EN as EL is to VL. But EM is leſs than EL, and the arch EN is greater than VL; therefore the ratio of EM to EN is leſs than the ratio of EL to VL. And theſe being contradictory, it follows, that the ratio of the fluxion of SE to the fluxion of *f*E is not greater than that of SE to SD. If the ratio of thoſe fluxions be leſs than that of SE to SD, let it be the ſame as that of SE to SK; produce DE and KE to *d* and I; and from *p* a point of the ſpiral within the angle *d*EI, let *pl* perpendicular on SE in *l* meet the right line EI in *u*; let S*p* meet the circle *f*E*b* in *n*, and S*m* be equal to S*p*. Then, ſince the fluxion of SE is to the fluxion of *f*E as SE is to SK, or as E*l* is to *ul*, E*m* is to the arch E*n* as E*l* is to *lu*. But E*m* is greater than E*l*; E*n* is leſs than its tangent E*x*, and therefore is leſs than *lu*; ſo that E*m* is to E*n* in a greater ratio than E*l* is to *lu*. And theſe being contradictory, it follows, that the fluxion of SE is to the fluxion of the arch *f*E as SE is to SD. From this it follows, converſely, that if SE be to SD as the velocity of M in the ray SE is to the velocity of N in the circle *f*E*b*, or (ſuppoſing S*f* to be the poſition of the ray SN when P ſets out from S the beginning of the ſpiral) as the ray SE is to the arch *f*E, then ſhall ED be the tangent of the ſpiral at E; and this coincides with the 21ſt propoſition of the treatiſe of ARCHIMEDES concerning that line.

203. *Caſe* 2. Suppoſe now the motion of M in the ray SE to be continually retarded while the motion of N in the arch NE*n* is uniform. It follows from the fourth axiom, that, if the motion of M be continued uniformly from the term when it comes to E, a leſs line than EM will be deſcribed by it in the ſame time N deſcribes an arch equal to EN; and, the fluxions of SE and *f*E being meaſured by the velocities of M and N at the term when they come to E, it follows, that EM is to EN in a greater ratio than that of the fluxion of SE to the fluxion of

*f*E.

*f*E. If the ratio of thoſe fluxions be greater than that of SE to SD, let it be the ſame as that of SE to S*k*, or that of EL to LV; and, P being a point of the curve within the angle *k*ED, as in the laſt article, EM ſhall be to EN in a greater ratio than EL is to LV; which is impoſſible, becauſe EM is leſs than EL, and EN is greater than LV. If the fluxion of SE be to the fluxion of *f*E in a leſs ratio than that of SE to SD, let their ratio be that of SE to SK, or (the conſtruction being the ſame as in the laſt article) that of E*l* to *lu*. It follows from the third axiom, that E*m* is to E*n* in this caſe in a leſs ratio than that of the velocity of M to the velocity of N at the term when theſe points come to E, or the ratio of the fluxion of SE to the fluxion of *f*E. Therefore E*m* is to E*n* in a leſs ratio than E*l* is to *lu*. But this is impoſſible, becauſe E*m* is greater than E*l*, and E*n* is leſs than *lu*. Therefore the fluxion of SE is to the fluxion of *f*E as SE is to SD.

FIG. 58. 204. *Caſe* 3. Let the motion of M in the ray SE be accelerated continually while the motion of N in the arch NE*n* is uniform, but ſo that the arch PE*p* may be ſtill on the ſame ſide of the tangent ED with S; and the fluxion of SE ſhall be to the fluxion of *f*E in a greater ratio than that of EM to the arch EN, by the ſecond axiom, but in a leſs ratio than that of E*m* to E*n*, by the firſt axiom. Let S*a* be perpendicular on the tangent in *a*; and, if the ratio of thoſe fluxions be that of SE to SK, which is leſs than the ratio of E*a* to S*a*, it follows from the 200th article, that a right line ST may be drawn from S meeting the tangent ED in T, the circle *f*E*b* in N, and the tangent of the arch EN in Z, ſo that TN may be to EZ as SE is to SK. Let this right line ST meet the curve FE in P; and, ſince TN is to the arch EN (which is leſs than the tangent EZ) in a greater ratio than SE is to SK, and therefore in a greater ratio than EM is to EN, it follows, that TN is greater than EM or PN, and that the point T is between P and S, againſt the ſuppoſition. If the fluxion of SE be to the fluxion of *f*E as SE is to S*k*, then, by the 199th article, a line S*t* may be drawn from S meeting the tangent DE produced beyond E in *t*, and the circle *f*E*b* in *n*, ſo that the ratio of *tn* to the chord E*n* may be the ſame as that of SE to S*k*, or of the fluxion of SE

to

to the fluxion of fE, which is leſs than the ratio of pn to the arch En. Therefore, ſuppoſing St to be ſuch a line, the ratio of tn to the arch En ſhall be leſs than the ratio of pn to the ſame arch; and tn ſhall be leſs than pn, ſo that the point t ſhall fall betwixt p and S, againſt the ſuppoſition. Therefore the fluxion of SE is to the fluxion of fE in a ratio that is neither greater nor leſs than that of SE to SD.

205. *Caſe* 4. Let the point S and the curve FPp be on diffe- FIG. 56.
rent ſides of the tangent ED; and, the motion of N being uniform, the motion of M muſt be perpetually accelerated, by the 201ſt article. In this caſe, the fluxion of SE is to the fluxion of fE in a greater ratio than that of EM to the arch EN, by the ſecond axiom; but in a leſs ratio than that of Em to the arch En, by the firſt axiom. If the ratio of theſe fluxions be that of SE to any line SK greater than SD, let Q be any point betwixt D and K; join EQ; and, by the 200dth article, a right line SR may be drawn from S to the right line EQ meeting it in R, the circle fEb in N, and its tangent in Z, ſo that RN ſhall be to EZ as SE is to SK, and, conſequently, in a greater ratio than that of EM to the arch EN. Suppoſe therefore SR to be ſuch a right line, and RN ſhall be greater than EM or PN; and, becauſe the ratio of RN to EZ continually increaſes while N moves towards E, it follows, that while N deſcribes the arch NE, the ratio of RN to the arch EN is greater than that of SE to SK, and that RN is greater than PN. But RN is leſs than TN; therefore the right line RE paſſes through the angle of contact PET formed by the curve PE and its tangent ET; which is abſurd, by art. 181. If the fluxion of SE was to the fluxion of fE as SE is to any line Sk leſs than SD; then, taking any point q betwixt D and k, joining qE, and producing it beyond E, a right line Sr might be drawn to it (by art. 199.) meeting it in r, and the circle Eb in n, ſo that rn might be to the chord En as SE is to Sk, or as the fluxion of SE is to the fluxion of fE, and, conſequently, in a leſs ratio than that of Em, or pn, to the arch En. Let Sr be ſuch a line, and rn ſhall be leſs than pn. Suppoſe the point n to move from n towards E, and the ratio of rn to En ſhall decreaſe continually by art. 199. and ſhall be leſs than that of SE to

Sk; therefore, during that time, *rn* is leſs than *pn*; but it is greater than *tn*; ſo that the right line E*r* muſt paſs through the angle of contact *p*E*t* formed by the curve E*p* and its tangent E*t*; which is abſurd, by art. 181. Therefore the fluxion of SE is to the fluxion of *f*E preciſely as SE is to SD. By joining theſe caſes together, the demonſtration is applicable when the motion of M is accelerated on one ſide of E and retarded on the other; and by the ninth and eleventh theorems, and the 60th article, it is rendred general. When the tangent ET is perpendicular to SE, and the velocity of N is given, the velocity of M at E and the fluxion of SE vaniſhes. When the ray SE touches the curve in E, and the velocity of M in the ray SE is given, the velocity of N at E and the fluxion of *f*E vaniſhes. This propoſition may be demonſtrated in another manner; but this ſeems to have the neareſt affinity to the method of ARCHIMEDES.

PROP. XVII.

206. *The ſame things being ſuppoſed as in the laſt propoſition, the fluxion of the curve* FE *is to the fluxion of the ray* SE *as the tangent* ED *is to the ray* SE.

FIG. 59.

For let SP meet the tangent always in T, and the velocities of the points P and T ſhall become equal at the term when they come together to E.

Firſt, let the curve FP*p* be convex towards S, and SP increaſe continually, as in the laſt article; then, the motion of N being ſuppoſed uniform, the motion of P in the curve ſhall be accelerated perpetually, by art. 201. The motion of the point T is alſo accelerated, and its velocity at the term when it comes to E is leſs than the velocity of P at any term after P paſſes E, as when it comes to *p*; becauſe a leſs line than E*t* would be deſcribed by the motion of T continued uniformly from E, in the ſame time that P would deſcribe a greater line than the arch E*p* (which exceeds E*t*) by its motion continued uniformly from

p,

p, by the firſt and ſecond axioms. Therefore, if the velocity of T at the term when it comes to E was greater than the velocity of P at that term, it might be equal to the velocity of P at the term when it comes to ſome point *h* betwixt E and *p*. Let *Sh* meet *Et* in *g*; and it follows from the ſecond axiom, that the point P would deſcribe a line greater than the arch *Eh* by its motion continued uniformly from *h*, in the time it deſcribes *Eh* with its accelerated motion; ſo that the point T would deſcribe a greater line than *Eg* by its motion continued uniformly from E in the ſame time that it deſcribes *Eg* with its accelerated motion, againſt the firſt axiom. In like manner it appears, that the velocity of T at the term when it comes to E cannot be leſs than the velocity of P at the ſame term. Therefore theſe velocities are equal, and, by prop. 15. the fluxion of the curve FP is to the fluxion of the ray SE as ED is to SE.

207. When the point P deſcribes a circle that paſſes through the point S, the motion of N being ſuppoſed uniform, the motion of P is alſo uniform. In this caſe, PE is always greater than TE, and *p*E leſs than *t*E. Therefore the velocities of P and T are equal at the term when they come to E, by the 53d article. When the point P deſcribes any other curve that is concave towards S, its motion is in ſome caſes accelerated and in others retarded. But it follows from what was ſhewn in the 54th, 55th and 56th articles, that the velocities of P and T are equal at the term when they come to E in thoſe caſes alſo. Therefore, in general, the fluxion of the curve FP is to the fluxion of the ray SE as ED is to SE.

208. Angles are meaſured by the arches which ſubtend them in equal circles; and in general they are to each other in the ratio compounded of the direct ratio of the arches which ſubtend them in any circles, and the inverſe ratio of the rays of theſe circles. The angular motion of the ray SN that generates any angle ASN is as the velocity of N in the circle EN when the ray SN is given, and is always as the velocity of N directly and the ray SN inverſely. The fluxion of the angle ASN is in the ſame ratio. FIG. 60.

209. Let the right lines CP, SP revolve about the given points C and S; and let their interſection P deſcribe any curve FEH

in the ſame plane with thoſe points; produce SC till it meet the curve in E, and when P comes to E, the angular motion of CP ſhall be to the angular motion of SP as SE is to CE. For let the circle EN deſcribed from the center S through E meet SP in N; and let the circle ER deſcribed from the center C through E meet CP in R; and let SD, CB be perpendicular on AE the tangent of the curve at E in D and B. Then the velocity of N ſhall be to the velocity of P at the term when they come together to E, as SD is to SE. The velocity of R ſhall be to the velocity of P at the ſame term, as CB is to CE; and SD is to SE as CB is to CE. Therefore the velocity of N is equal to the velocity of R at that term; and the angular motion of SP is to the angular motion of CP, when P comes to E, as CE is to SE, by the laſt article.

PROP. XVIII.

210. *Let* P, *the interſection of the right lines* CP *and* SP *revolving about the given points* C *and* S, *deſcribe the curve* FEH; *let* AO *touch this curve in* E; *conſtitute the angle* SET *equal to* CEA *ſo that* ET *and* EA *may lie different ways from* SE *and* CE; *and let* ET *meet* CS *in* T. *Then, if* CA, Sa *be right lines given in poſition, the fluxion of the angle* ACP *ſhall be to the fluxion of a* SP, *when* P *comes to* E, *as* ST *is to* CT.

Fig. 61. & 62.

From the centers C and S deſcribe through the point E the circles ER and EN, meeting CP and SP in R and N reſpectively; and let CB and SD be perpendicular on the tangent AO in B and D. Let CK parallel to ET meet SE, produced if neceſſary, in K; and let EQ be perpendicular on CK in Q. The angle QKE is equal to SET or CEB by the conſtruction; and therefore the triangle QKE is ſimilar to CEB, ſo that CE is to CB as EK is to EQ. The triangles SED, CEQ are alſo ſimilar, and SE is to SD as CE is to EQ. But when P and R come

come to E, the velocity of P is to the velocity of R as CE is to CB, (by prop. 15. & 17.) that is, as EK is to EQ. The velocity of P is to the velocity of N at the ſame term as SE is to SD, or as CE is to EQ. Therefore the velocity of R is to the velocity of N, when they come together to E, as CE is to EK. This ratio compounded with the inverſe ratio of the ray CR to the ray SN, or with the ratio of SE to CE, gives the ratio of SE to EK, or (becauſe ET is parallel to CK) of ST to CT; which therefore (by art. 208.) is the ratio of the angular velocity of CP about C to the angular velocity of SP about S when P comes to E, or the ratio of the fluxion of the angle ASE to the fluxion of the angle *a*SE. When the angle CEA is equal to CSE, the point K falls on S, the angular motions of CE and SE are equal, and the fluxion of the angle ACE is equal to the fluxion of *a*SE. When the tangent AE paſſes through S, the point T falls on C; and in this caſe, the angular velocity of the ray CE being given, that of the ray SE vaniſhes. The point E may coincide with S, but we reſerve this caſe till we come to treat of the curvature of lines. The fluxion of the angle CPS is equal to the ſum or difference of the fluxions of the angles ACP, *a*SP, and is to the fluxion of the angle ACP or the fluxion of ASP, when P comes to E, as CS is to ST or to CT. When the points C and S are not in the plane of the curve EEH, the ratio of the angular velocities of the rays CE and SE may be deduced from this propoſition. For if V be any point in the tangent AE, and the right line V*ſ* equal to VS be drawn in the plane CVE conſtituting the angle *ſ*VE equal to SVE, the angular velocity of *ſ*E about *ſ* ſhall be equal to the angular velocity of SE about E. Fig. 63.

211. The preceeding propoſitions ſhew the analogy there is betwixt the method of Fluxions and the method of Tangents, and ſerve for determining the tangents from the fluxions of lines and angles, as well as for finding theſe fluxions from the tangents. Beſides theſe general theorems, there are many particular propoſitions that are often of uſe in determining the tangents of curve lines; ſome of which we ſhall briefly deſcribe. Let S be a given point in the plane of the curve ALB, LP a tangent at L any point of the curve, SP perpendicular on the Fig. 64.

tan-

tangent in P; and let P be always found in the curve DPE. Let PT conſtitute with SP an angle SPT equal to the angle SLP, on the ſame ſide of SP as LP is of SL; and PT ſhall be the tangent of the curve DPE at P. For, firſt, let the arch L*l* of the curve ALB be concave towards S, and the rays drawn from S to the arch L*l* decreaſe from SL to S*l*, and let S*p* be perpendicular on *lp* the tangent at *l* in *p*; join P*p*, and let *pl* meet PL in R: then, becauſe the angles SPR, S*p*R are right, the angle SP*p* is equal to SR*p*; and the angle TP*p* is equal to the difference betwixt the angles SLP and SR*p*. The angle SLP is equal to the difference of the angles SRP and RSL; the angle SR*p* is equal to the difference of the angles SRP and PR*p* or PS*p*: therefore the angle TP*p* is equal to the difference of the angles PS*p* (or PR*p*) and RSL. But, by ſuppoſing *l* to move towards L, the angles PS*p* and RSL and their difference may become leſs than any given angle. Therefore, while *p* moves towards P, the angle TP*p* may become leſs than any given angle. From which it follows, that no right line can be drawn through the angle of contact formed by the right line PT and the arch P*p*. Therefore PT is the tangent at P. When ALB is a parabola, and S its focus, DPE is a right line; but, when ALB is any other curve, DPE is a curve line. In the parabola, SR*p* is an angle always of the ſame magnitude wherever the point *l* be taken, the point L being given; and PS*p* is always equal to RSL or one half of LS*l*. In other caſes, according as the angle PS*p* is greater or leſs than RSL, (or the fluxion of the angle ASP is greater or leſs than one half of the fluxion of the angle ASL,) the angle SPT is greater or leſs than SP*p*, and the arch P*p* is concave or convex towards S.

FIG. 65. 212. When the arch L*l* is convex towards S, the conſtruction in other reſpects being the ſame, produce *p*P to p; and the angle SPp ſhall be equal to SR*p*. Therefore the angle TPp is equal to the difference of the angles SR*p* and SLP, which in this caſe is equal to the ſum of PR*p*, or PS*p*, and RSL; and, ſince this ſum may become leſs than any given angle when *l* moves towards L, it follows, that PT is the tangent of the curve DPE at P. In this caſe, the arch P*p* is concave towards S. Let ST be perpendicular on PT in T, and let the point T be always

always found in the curve FT; then the tangent of this curve at T shall constitute an angle with ST equal to SPT or SLP. There is a series of curves which may be conceived to be derived from each other in this manner: The tangents form always equal angles with the rays drawn from S at the corresponding points of the curves, and the fluxions of the curves are to the fluxions of the rays at these points in the same ratio in them all.

213. Let any curvilineal figure CEH be the base of any conical surface that has its vertex in V, and ET be the tangent of the arch EH in E. Let *ceh* be a section of this conical figure made by any plane; and let *et* be the common section of that plane and the plane VET: then shall *et* be the tangent of the arch *eh*. For, if *et* meet the arch *eh* in any point besides *e*, it is manifest that ET must meet the arch EH in some other point besides E; and if any right line, as *ex*, can be drawn through the angle of contact *het* formed by the arch *eh* and the right line *et*, let the common section of the planes *Vex*, CEH be EX; and EX shall pass through the angle of contact HET, against the supposition. It is obvious, that the arch *eh* and its tangent *et* are the shadows of the arch EH and tangent ET formed by rays issuing from V upon the plane *ceh*. FIG. 66.

214. Let the right lines CH, SK revolving about the poles C and S by their intersection P describe the curve APS that passes through S; and let S*k* be the situation of the right line SK when CH passes through S; then shall S*k* be the tangent at S. FIG. 67.

CHAP. VIII.

Of the Fluxions of curve Surfaces.

215. ARCHIMEDES establishes his theorems concerning curve surfaces upon this principle, That, when two curve surfaces have their concavities turned the same way and have the same terms, that which includes the other is the greater surface: and this axiom is sufficient for demonstrating the cases that were considered by him. But, because it cannot be applied

for

for comparing ſurfaces that are generated by curves which are convex towards the axis about which the figure is ſuppoſed to revolve, we ſhall make uſe of the following principle in place of it, that is more general, and ſeems to be no leſs evident. Let CEH be any arch of a curve; and let BG the axis be divided by a continual biſection into any number of equal parts BK, KD, DL, LG. Let the ordinates BC, KM, DE, LN, GH meet the curve in the points C, M, E, N, H; and let the tangents at C, M, E, H form the circumſcribed figure CQRSVH. Then, by continuing to biſect the parts of the axis BG, and ſuppoſing the ordinates, chords and tangents to be drawn as before, the perimeters of the circumſcribed and inſcribed figures CQRSVH, CMNEH ſhall approach continually to the arch CEH; and, ſuppoſing the whole figure to revolve about the axis BG, the ſurfaces deſcribed by thoſe perimeters ſhall approach to the ſurface deſcribed by the arch CEH, ſo that the differences betwixt them may become leſs than any given ſurface. The axiom cited from ARCHIMEDES in art. 183. may be deduced from this principle.

Fig. 68. n. 1, 2.

LEMMA VIII.

216. *Let the right line* HC *produced meet the axis* AG *in* A, *and the ſurface deſcribed by* HC *revolving about* AG *ſhall be equal to the area of a circle the radius of which is a mean proportional betwixt* 2HC *and the right line* DE *that biſects* HC *in* E, *and is perpendicular to the axis in* D.

Fig. 69.

Let HG and CB perpendicular to the axis meet it in G and B. By what was ſhewn after ARCHIMEDES, in the Introduction, pag. 11. the ſurface deſcribed by AH is equal to a circle the radius of which is a mean proportional betwixt AH and HG; and the ſurface deſcribed by AC is equal to the area of a circle that has its radius a mean proportional betwixt AC and BC. Therefore theſe ſurfaces are to each other as the rectangle AHG is to the rectangle ACB, or as the ſquare of AH is to the ſquare of AC;

Pl. VIII. Pa. 200.

Fig. 57.

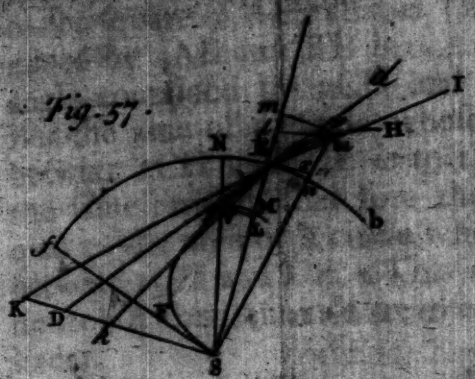


Fig. 58.

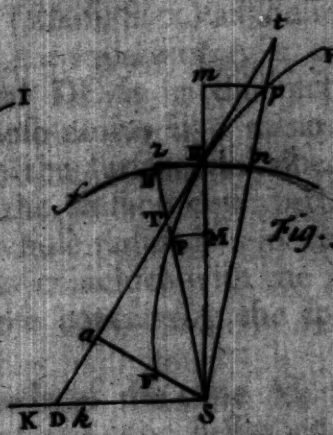


Fig. 59 N. 1.

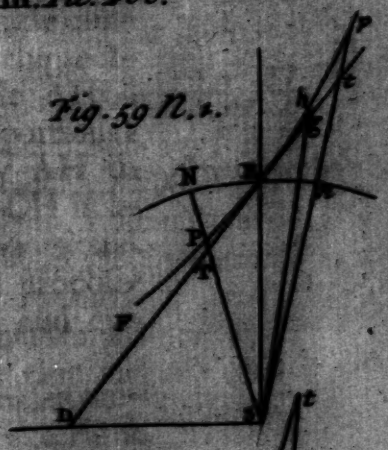


Fig. 59. N. 2

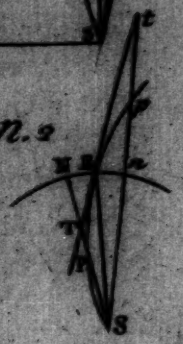


Fig. 61.

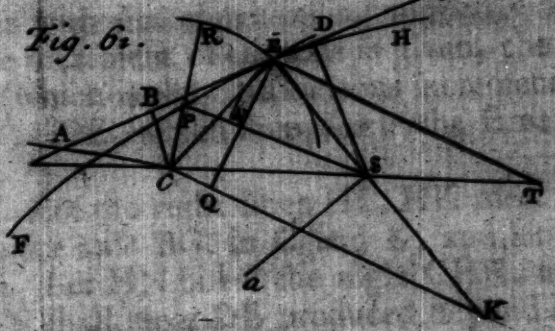


Fig. 62.

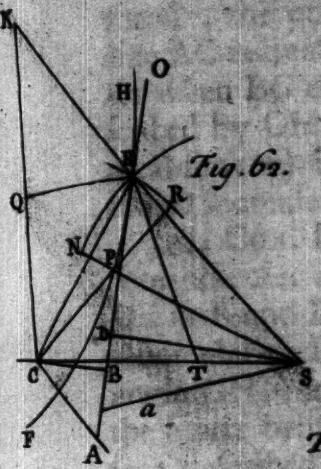


Fig. 63.

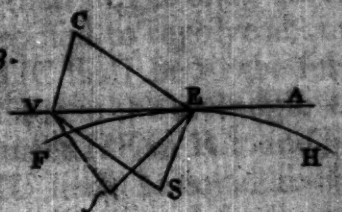


Fig. 60.

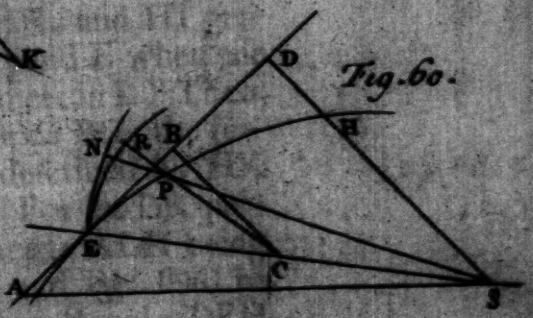


Fig. 65.

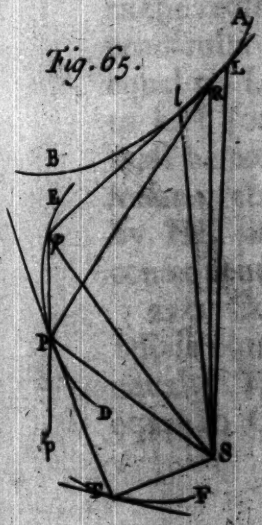


Fig. 64.

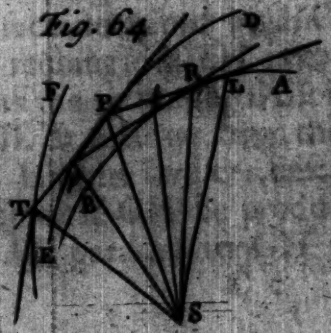
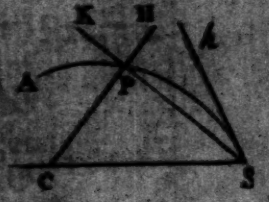
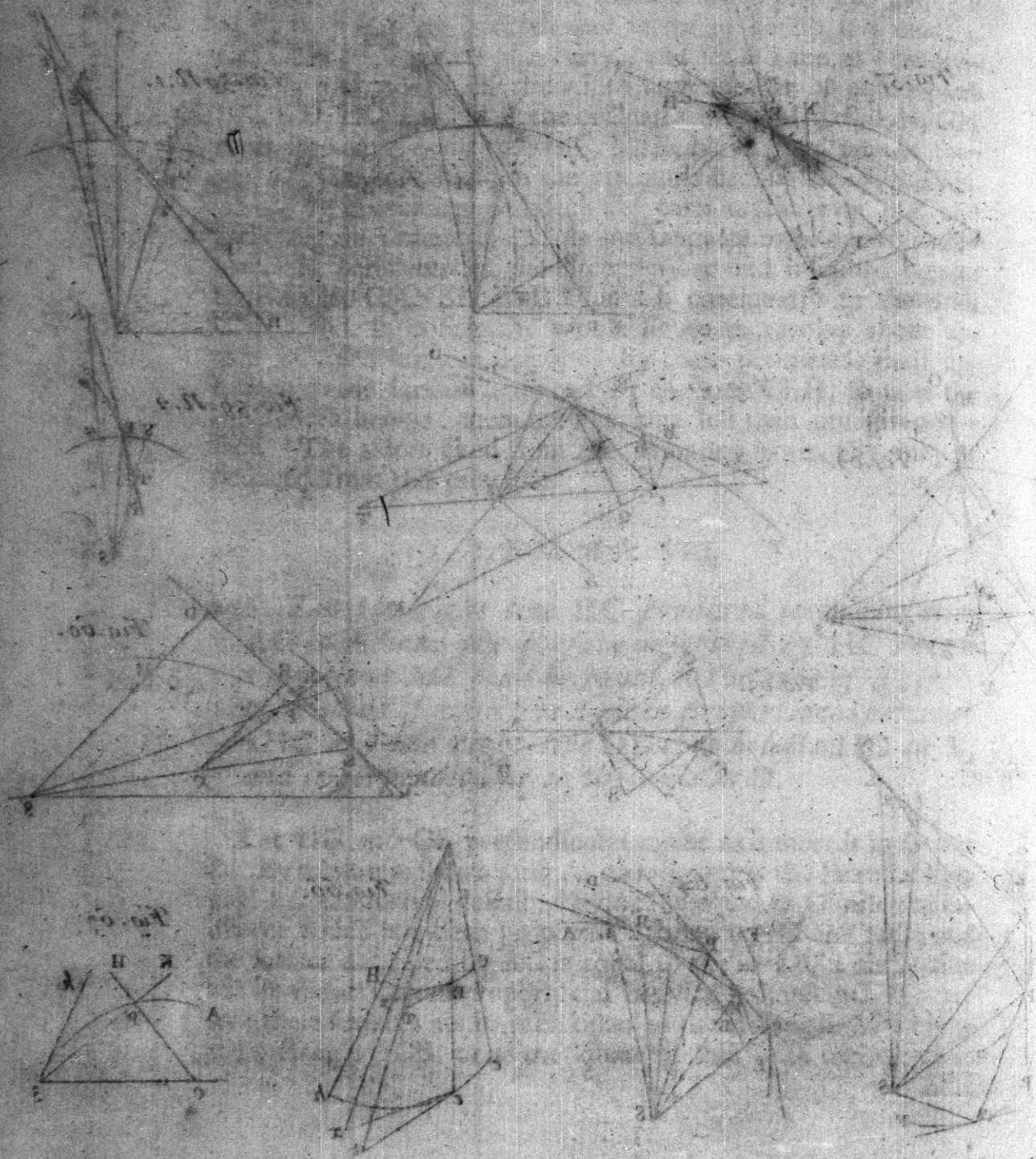


Fig. 66.

Fig. 67.





AC; and the ſurface deſcribed by CH is to the ſurface deſcribed by AH, as the difference of the ſquares of AH and AC is to the ſquare of AH, that is, (Elem. 8. 2.) as the rectangle contained by 4AE and EH is to the ſquare of AH, or as the rectangle contained by 2CH and DE is to the rectangle AHG. But the area of a circle whoſe radius is a mean proportional betwixt 2CH and DE is to the ſurface deſcribed by AH in the ſame ratio. Therefore the ſurface deſcribed by CH is equal to the area of a circle whoſe radius is a mean proportional betwixt 2CH and DE. This coincides with the 16th propoſition of ARCHIMEDES's treatiſe concerning the ſphere and cylinder.

217. COR. I. When the axis of the cone AB increaſes uniformly, the convex ſurface of the cone deſcribed by the right line AC increaſes with a motion that is continually accelerated; for when BG the increment of the axis is given, the ſurface deſcribed by CH (which is the ſimultaneous increment of the conical ſurface) is as DE, which increaſes in the ſame proportion as AD increaſes.

218. COR. II. Let the right lines HC, HK meet BC that is perpendicular to the axis BG, in C and K, and let them meet DE in E and L. Let HN biſect the angle CHK, and HI perpendicular to HN ſhall meet DE produced beyond E when the angle GHN is leſs than a right one; let them meet in I. Then, the figure being ſuppoſed to revolve about the axis BG, the ſurface deſcribed by HC ſhall be to the ſurface deſcribed by HK as the rectangle DEI is to the rectangle DLI. For let LR perpendicular to HN meet HE in R; and, the angle RLH being equal to LRH, HL ſhall be equal to HR; and EH ſhall be to LH (or RH) as EI is to LI; ſo that the rectangle DEH ſhall be to the rectangle DLH as DEI is to DLI. But, by this lemma, the ſurface deſcribed by HC is to the ſurface deſcribed by HK as the rectangle DEH is to the rectangle DLH, and, conſequently, as the rectangle DEI is to DLI.

219. COR. III. It follows from the laſt corollary, that, when DE is equal to LI, the ſurfaces deſcribed by HC and HK are equal. Therefore, if GH be produced beyond H till HM be equal to GH, AF parallel to GH meet HK in F, FM meet FIG. 70.

GO perpendicular to HN in O, and BC produced paſs through O; the ſurfaces deſcribed by HC and HK ſhall be equal. For, in this caſe, DE, HI and OM interſect each other in the point I that is middle betwixt O and M; and, becauſe GH is equal to HM, DE is equal to LI.

FIG. 71. 220. COR. IV. Let CV be perpendicular on HK in V; produce it till it meet DE in S: and, when ES is either equal to ED, or leſs than it, the ſurface deſcribed by HC is greater than the ſurface deſcribed by HK. For, in the triangles ECS, EHI, the ſide EC is equal to EH, the angle CES is equal to HEI, but the angle ECS is greater than EHI, (becauſe the angle ECS is equal to the right angle CVH added to CHV, whereas EHI is only equal to the right angle NHI added to CHN;) and therefore ES is greater than EI. Therefore, when ES is either equal to ED, or leſs than it, DE is greater than EI, the rectangle DEI is greater than DLI, and the ſurface deſcribed by HC is greater than the ſurface deſcribed by HK.

221. COR. V. The ſame things being ſuppoſed as in the laſt corollary, let *k* be any point betwixt C and K, and the ſurface deſcribed by HC ſhall be greater than the ſurface deſcribed by H*k*, which is itſelf greater than that which is deſcribed by HK. For, let C*u* perpendicular on H*k* in *u* be produced till it meet DE in *ſ*, and, E*ſ* being leſs than ES, (which is ſuppoſed to be either equal to ED or leſs,) it follows, that the ſurface deſcribed by HC is greater than that which is deſcribed by H*k*. Let H*k* meet EL in *e*, and *ko* perpendicular on HK meet *e*D in ſ, and *e*ſ being leſs than *e*D, the ſurface deſcribed by H*k* muſt be greater than that which is deſcribed by HK, by the laſt corollary.

FIG. 72. 222. COR. VI. Let *h* be a point on the line HC betwixt H and C, or any where within the triangle CHK; let *hk* parallel to HK meet CK in *k*, and *hc* meet CK in any point *c* betwixt C and *k*; then the ſurface deſcribed by *bc* ſhall be greater than that which is deſcribed by *hk*. For let *hg* parallel to HG meet the axis in *h*, let *de* parallel to *hg* biſect B*h* in *d*, and meet *hc*, *hk* in *e* and *l*; and it is manifeſt, that if *cu* be perpendicular to *hk*, or parallel to CV, it ſhall meet *ed* in ſome point *ſ* betwixt *e* and *d*.

223. COR.

223. Cor. VII. It appears in the ſame manner, that, when *h* is any point within the triangle CHK, *c* a point on CK, *k* a point on *c*K, if *ch* and *kh* produced beyond *h* meet CH in *r* and *n*, ſo that the angles C*rh*, C*nh* be each leſs than the angle CHK, then the ſurface deſcribed by *hc* ſhall be greater than the ſurface deſcribed by *hk*. Fig. 73.

224. Cor. VIII. Let HEC be now an arch of a continued curvature convex towards the axis BG; let the ordinates from BC to GH increaſe continually, and let the ordinate DE biſect BG; let HT the tangent at H meet BC in T, and ſuppoſe that T*u* perpendicular on CQ the tangent at C meets DE in *ſ*, ſo that E*ſ* is either equal to the ordinate DE, or leſs than it. Then the ſurface deſcribed by the arch HEC revolving about the axis BG ſhall be leſs than the ſurface deſcribed by the tangent HT revolving about the ſame axis. For let H*k* parallel to the tangent CQ meet BC in *k*, and T*u* in V, the angle TVH being a right one, and the arch HEC being within the triangle TH*k*, it appears, that, if the chord HE be produced till it meet CT in *r*, the ſurface deſcribed by HT ſhall be greater than that which is deſcribed by H*r*, by cor. 5. and the ſurface deſcribed by E*r* greater than that which is deſcribed by EC, by cor. 7. Therefore the ſurface deſcribed by the tangent HT is greater than the ſum of the ſurfaces which are deſcribed by the chords HE and EC. In like manner, the parts of the axis BD, DG being biſected by the ordinates KM, LN, if the chords HN, NE, EM be produced till they meet CT in *u*, *x* and *z*, the ſurfaces deſcribed by N*u*, E*x*, M*z* ſhall exceed the ſurfaces deſcribed by N*x*, E*z*, MC reſpectively, by cor. 7. and therefore the ſurface deſcribed by HT being greater than that which is deſcribed by H*u*, which exceeds that deſcribed by HN*x*; and, this ſurface being greater than that which is deſcribed by HNE*z*, which exceeds the ſurface deſcribed by HNEMC, it follows, that the ſurface deſcribed by HT is greater than that which is deſcribed by HNEMC. In general it appears, that the ſurface deſcribed by the tangent HT is greater than the ſurface which is deſcribed by the perimeter of any rectilineal figure inſcribed in the arch HEC; and it may be ſhewn in the ſame manner, that it is greater than the ſurface deſcribed by the perimeter Fig. 68. n. 1.

rimeter of any rectilineal figure circumſcribed about the arch formed by its tangents. Therefore the ſurface deſcribed by the tangent HT is greater than that which is deſcribed by the arch HEC, by art. 215.

225. Cor. IX. Join GC, and, if the angle GCQ which it forms with the tangent at C be either a right angle, or greater than a right one, and CT be not leſs than TB, the ſurface deſcribed by the tangent HT ſhall be greater than that which is deſcribed by the arch HEC: for, in this caſe, E*f* is either equal to ED, or leſs than it. It appears therefore, that, when the curve is convex towards the axis, and the ordinates increaſe while the axis increaſes, an arch of the curve, as CH, may be taken of a finite magnitude ſo that the ſurface deſcribed by it ſhall be leſs than the ſurface which is deſcribed by HT the tangent at H terminated in T by the ordinate from C. If the tangent CQ meet GH in Q, it is evident, that the ſurface deſcribed by CMENH the perimeter of any rectilineal figure inſcribed in the arch CEH is always greater than the ſurface deſcribed by the tangent CQ. Therefore the ſurface deſcribed by the arch CEH exceeds that which is deſcribed by the tangent CQ, by art. 215.

226. Cor. X. Let the curve FEH be convex towards the axis AG, and the ordinates *pm* increaſe while the axis A*p* increaſes; then, if the axis A*p* increaſe uniformly, the ſurface deſcribed by the curve F*m* ſhall increaſe with an accelerated motion. For, while the axis acquires the equal augments BD, DG, let the curve acquire the augments CME and ENH; and it is manifeſt, that the ſurface deſcribed by the perimeter of any rectilineal figure inſcribed in the arch HNE is greater than the ſurface deſcribed by the perimeter of a rectilineal figure inſcribed by a ſimilar conſtruction in the arch EMC. Therefore the ſurface deſcribed by the arch EH is greater than that which is deſcribed by the arch CE. Let CQ meet DE in *q*, and the tangent at H meet DE in *t*; then, the ſurface deſcribed by the arch CE being greater than that which is deſcribed by the tangent C*q*, and the ſurface deſcribed by the arch EH being leſs than that which is deſcribed by the tangent H*t* when CEH is diminiſhed as in cor. 9. it follows, that the motion with

with which the ſurface deſcribed by F*m* flows, while *m* deſcribes the arch CEH, is accelerated in a continued manner.

227. Cor. XI. The reſt remaining as in the laſt corollary, when the arch CEH is concave towards the axis BG, BD may be taken ſo ſmall that the ſurface deſcribed by the tangent ET ſhall be leſs than the ſurface deſcribed by the chord CE, (by cor. 4.) and therefore leſs than the ſurface deſcribed by the arch CE, which exceeds that which is deſcribed by the chord CE, by art. 215. The ſurface deſcribed by the arch EH is leſs than that which is deſcribed by the tangent E*t*, by the ſame article. In this caſe, when the axis increaſes uniformly, the motion with which the ſurface deſcribed by the curve flows may be uniform, accelerated or retarded. FIG. 75.

LEMMA IX.

228. *Let* DE *and* G*t* *perpendicular to the axis* a G *meet* a E *in* E *and* *t*; *and, in the ſame time that the motion with which the axis* a D *flows continued uniformly generates* DG, *let the motion with which the conical ſurface deſcribed by* a E *flows continued uniformly generate a ſpace equal to* R; *then ſhall the ſpace* R *be equal to the area of a circle the radius of which is a mean proportional betwixt* DE *and* 2E*t*. FIG. 74.

For, if the ſpace R be greater than ſuch a circle, let A*d* be greater than AD in the ſame ratio; and, if *do* parallel to DE meet E*t* in *o*, R ſhall be equal to a circle whoſe radius is a mean proportional betwixt *do* and 2E*t*; becauſe the area of this circle is to the area of a circle whoſe radius is a mean proportional betwixt DE and 2E*t* as *do* is to DE, or as A*d* is to AD. The ſpace which is generated by the motion with which the ſurface a E*t* flows continued uniformly, in the time that the axis a D by flowing uniformly acquires the augment D*d*, is to the ſpace R as DG is to D*d*, (by theor. 2.) or as E*t* is to E*o*, and therefore is equal to the area of a circle whoſe radius is a mean proportional betwixt *do* and 2E*o*; but the-

the area of this circle is greater than the ſurface deſcribed by the right line Et, by lemma 8. and the conical ſurface *a*E*e* increaſes with an accelerated motion when the axis increaſes uniformly, by art. 217. Therefore a greater ſpace is generated in the ſame time by a motion continued uniformly, than when the ſame motion is continually accelerated, againſt the firſt axiom. In the ſame manner it appears from the ſecond axiom, that the ſpace R is not leſs than the area of a circle the radius of which is a mean proportional betwixt DE and 2E*t*.

PROP. XIX.

229. *Let* DE *and* GH *perpendicular to the axis meet the curve in* E *and* H, *and let* GH *meet the tangent at* E *in t; then, the fluxion of the axis being repreſented by* DG, *the fluxion of the ſurface deſcribed by the curve* FE *ſhall be accurately meaſured by the area of a circle the radius of which is a mean proportional betwixt* DE *and* 2E*t*.

FIG. 74.

Let the ordinate PM always meet the curve in M and the tangent *a*E in N; and the motions with which the ſurfaces F*fm*M, *a*N*n* flow ſhall be equal at the term when M and N come together to E, or the fluxion of the ſurface F*fe*E ſhall be equal to the fluxion of the ſurface *a*E*e*. For, the conſtruction being the ſame as in the 226th and 228th articles, firſt let the arch CEH be convex towards the axis BG, and the ſurface F*fm*M ſhall flow with an accelerated motion while P and M deſcribe BG and CH, by art. 226. Suppoſe, firſt, BD to be ſo ſmall, according to the 225th article, that the perpendicular from T upon the tangent at C may meet DE in D, or in ſome point betwixt D and E, and the ſurface deſcribed by ET ſhall be greater than the ſurface deſcribed by the arch EC. But the latter ſurface is greater than the ſpace which would be generated by the motion with which the ſurface F*fc*C flows continued uniformly, in the time P deſcribes BD with an uniform motion, (by the firſt axiom;) and the former is leſs than the

ſpace

ſpace which would be generated in an equal time by the motion with which the ſurface *a*E*e* flows continued uniformly, by the ſecond axiom. Therefore the motion with which the ſurface *a*E*e* flows, is greater than that with which the ſurface F*fc*C flows; and it is manifeſt from the ſame axioms, that it is leſs than the motion with which the ſurface F*fb*H flows; ſo that it muſt be equal to the motion with which the ſurface F*fm*M flows when M comes to ſome intermediate point of the arch CH. If it be ſaid to be equal to the motion with which the ſurface F*fx*X flows, X being ſome point betwixt E and H, let *d*X perpendicular to the axis in *d* meet the tangent E*t* in *o*. The motion with which the ſurface F*fx*X flows continued uniformly generates a ſpace greater than the ſurface E*ex*X in the time P deſcribes D*d*, by ax. 1. The ſurface E*ex*X is greater than the ſurface deſcribed by the right line E*o* revolving about D*d*, (by art. 225.) Therefore the motion with which the ſurface *a*E*e* flows continued uniformly, would generate a ſpace greater than the ſurface deſcribed by E*o*, in the time P deſcribes D*d* uniformly: But a leſs ſpace than this ſurface would be deſcribed in that time by the motion with which *a*E*e* flows continued uniformly, by the firſt axiom; And theſe are contradictory. Therefore the motion with which the ſurface *a*E*e* flows, is not greater than the motion with which the ſurface F*fe*E flows. Nor can it be leſs. For, if it be ſaid to be equal to the motion with which the ſurface F*fy*Y flows, Y being any point of the curve betwixt E and C, let Yd meet the axis in d, and the tangent ET in o. Then, the ſurface deſcribed by Eo about the axis BD being greater than the ſurface E*e*y*Y*, (by art. 225.) and this ſurface being greater than the ſpace which would be generated by the motion with which F*fy*Y flows continued uniformly in the time P deſcribes dD uniformly, (by ax. 1.) it follows, that the ſpace which would be generated in that time by the motion with which *a*E*e* flows is leſs than the ſurface deſcribed by Eo: But it is greater than that ſurface by the ſecond axiom: And theſe are contradictory. Therefore the motions with which the ſurfaces F*fe*E, *a*E*e* flow are equal. By the laſt lemma, the fluxion of the axis AD being repreſented by DG, the fluxion of the ſurface *a*E*e* is meaſured by the area of a circle the radius of

which

which is a mean proportional betwixt DE and 2E*t*. Let DG be increaſed in any proportion, and E*t* ſhall be increaſed in the ſame proportion; ſo that, if DG repreſent the fluxion of AD, the fluxion of the ſurface F*fe*E ſhall be accurately meaſured by the area of a circle the radius of which is a mean proportional betwixt DE and 2E*t*.

FIG. 75. 230. Let the arch CEH be now concave towards the axis; and BD may be taken in this caſe ſo ſmall that the ſurface deſcribed by the arch CE ſhall be greater than the ſurface deſcribed by the tangent ET, (by art. 227.) but the ſurface deſcribed by E*t* is always greater than that which is deſcribed by the arch EH. When the axis increaſes uniformly, the motion with which the conical ſurface deſcribed by *a*N flows is continually accelerated, by art. 219. and, whether the motion with which the ſurface F*fm*M flows be uniform, accelerated or retarded, it follows from what was demonſtrated in the 53d, 54th and 56th articles, that the motions with which the ſurfaces F*fe*E, *a*E*e* flow are equal. Therefore, the fluxion of the axis AD being repreſented by DG, the fluxion of the ſurface F*fe*E is meaſured by the area of a circle whoſe radius is a mean proportional betwixt DE and 2E*t*. The demonſtration is rendred general by the propoſitions which have been ſo often cited from the firſt chapter.

FIG. 76. 231. COR. I. Let EO perpendicular to the tangent E*t* meet the axis in O; and, becauſe EO is to DE as E*t* is to DG, it follows, that the fluxion of the ſurface F*fe*E is meaſured by the area of a circle whoſe radius is a mean proportional betwixt 2EO and DG, when the fluxion of the axis is repreſented by DG. Therefore, when FEH is an arch of a circle whoſe diameter AB coincides with the axis of motion, the perpendicular EO being invariable, the ſpherical ſurface AE*e* flows uniformly when the axis AD flows uniformly; and, in the ſame time that the uniform motion with which the axis flows generates AD, the uniform motion with which the ſpherical ſurface flows generates a ſurface equal to the area of a circle whoſe radius is a mean proportional betwixt AD and AB; that is, the ſpherical ſurface AE*e* is equal to the area of a circle deſcribed with the chord AE; and the whole ſurface of the ſphere is equal to a circle whoſe radius is AB the diameter of the ſphere, and therefore

is

is quadruple of a great circle of the ſphere, as was demonſtrated after ARCHIMEDES in the Introduction.

232. COR. II. But if the circle be ſuppoſed to revolve about a chord AB that is not a diameter, let VK perpendicular from the center V upon AB meet it in K; let L*l* be the diameter parallel to AB, A*a* and B*b* perpendicular to L*l* meet the circumference in *a* and *b*; produce DE till it meet the circumference again in *e*, and let it meet AR the tangent at A in R. Suppoſe the right line R to be the radius of a circle whoſe area is A. Then, if R be a mean proportional betwixt 2VK and the exceſs of AR above the arch AE when E is taken upon the leſſer arch ſubtended by AB, the area A ſhall be equal to the ſurface deſcribed by the arch AE by revolving about the axis AB; and, if R be a mean proportional betwixt 2VK and the ſum of the right line AR added to the arch AE or *ae*, the area A ſhall be equal to the ſurface deſcribed by the arch *ae* about AB. But, if E be taken betwixt A and L; then, according as R is a mean proportional betwixt 2VK and the exceſs of the arch AE above AR, or betwixt 2VK and the ſum of AE and AR, the area A is equal to the ſurface deſcribed by AE, or that which is deſcribed by *ae*. For, let G*t* produced meet AR in *r*, and let E*e* meet L*l* in *d*; and, ſince R*r* is to DG as AR is to AD, or VA to VK, and DG is to E*t* as *d*E is to VE, it follows, that R*r* is to E*t* as *d*E is to VK; and that VK is to DE as E*t* is to the difference of R*r* and E*t*, but that VK is to D*e* as E*t* is to the ſum of R*r* and E*t*. Therefore the fluxion of the ſurface deſcribed by AE is meaſured by a circle whoſe radius is a mean proportional betwixt 2VK and the difference of R*r* and E*t*, and the fluxion of the ſurface deſcribed by the arch *ae* is meaſured by a circle whoſe radius is a mean proportional betwixt 2VK and the ſum of R*r* and E*t*. But R*r* meaſures the fluxion of AR, and E*t* meaſures the fluxion of the arch AE or *ae*. Therefore, according as R is a mean proportional betwixt 2VK and the difference of AR and AE, or betwixt 2VK and the ſum of AR and AE, the area A is equal to the ſurface deſcribed by the arch AE, or that which is deſcribed by *ae*. When E is upon the leſſer arch ſubtended by the chord AB, the ſum of the ſurfaces deſcribed by AE and *ae* is equal to the area of a circle

Fig. 76.

whose radius is a mean proportional betwixt 2VK and 2AR, or betwixt the diameter and 2AD; but when E is betwixt A and L, the sum of the surfaces described by AE and *ae* is equal to a circle whose radius is a mean proportional betwixt 4VK and the arch AE. Let *b*B produced meet AR in M, and the surface described by the whole circumference of the circle, by revolving about the chord AB, is equal to a circle the radius of which is a mean proportional betwixt 4VK and the sum of the right line AM and arch AL*a*. The difference of the surfaces described by the arches A*ab*B and AEB is equal to a circle whose radius is a mean proportional betwixt 2VK and the whole
FIG. 77. circumference. When the axis AB does not meet the circle, let L*l* be the diameter parallel to AB, and let LA, *l*B, VK, *d*D be perpendicular to the axis in A, B, K and D. Let E be in the upper, and *e* in the lower semicircle; and, according as the square of R is equal to the sum or difference of the rectangles contained by 2VK and the arch LE, and by 2VL and AD, the area A is equal to the surface described by LE, or that which is described by L*e*. The surface described by the arch EL*e* is equal to a circle the radius of which is a mean proportional betwixt this arch and 2VK; and the surface described by the whole circumference is equal to a circle the radius of which is a mean proportional betwixt the circumference and 2VK. When the axis AB touches the circle, this radius is a mean proportional betwixt the circumference and diameter.

FIG. 78. n. 1, 2. 233. COR. III. In general, let F*f* be an arch of any curve line, let *ab* parallel to AB meet DE in *d*; and the area of a circle whose radius is a mean proportional betwixt 2D*d* and the arch F*f* shall be equal to the sum of the surface described by the arch F*f* revolving about the axis AB added to the surface described by the same arch revolving about *ab*, when the arch F*f* is betwixt AB and *ab*, but equal to the difference of those surfaces when AB and *ab* are on the same side of the arch F*f*; because, in the former case, the rectangle contained by D*d* and E*t* is equal to the sum of the rectangles DE*t*, *d*E*t*, and in the latter case it is equal to the difference of those rectangles: and therefore the sum of the fluxion of the surface described by FE about AB, added to the fluxion of the surface described by FE about

Pl. IX. Pa. 210.

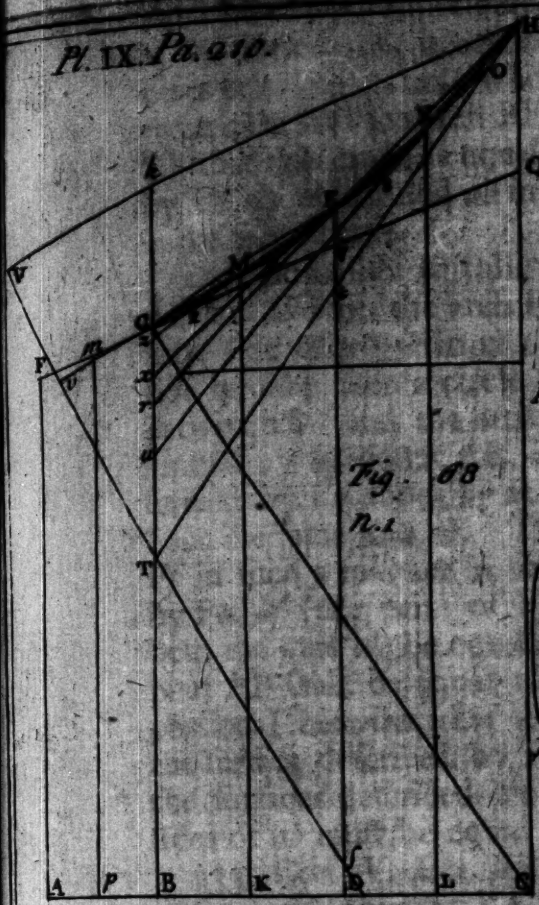


Fig. 68 n. 1

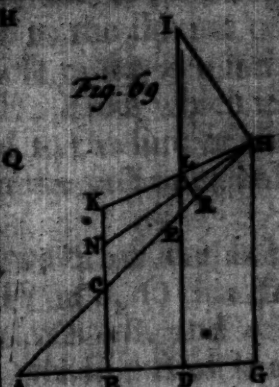


Fig. 69

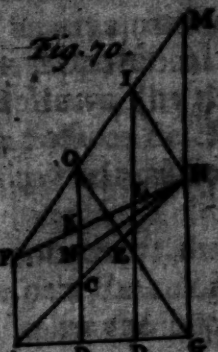


Fig. 70.

Fig. 71.

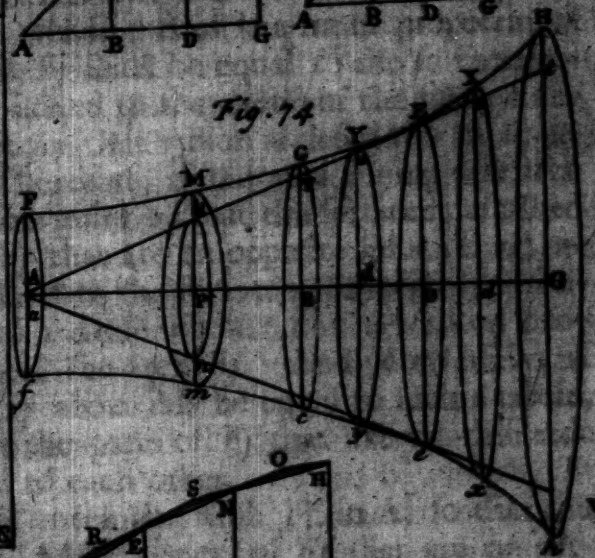
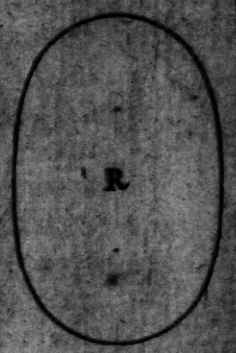


Fig. 74



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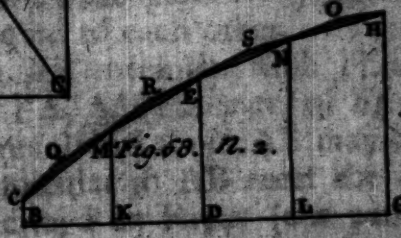


Fig. 68. n. 2.

Fig. 72.

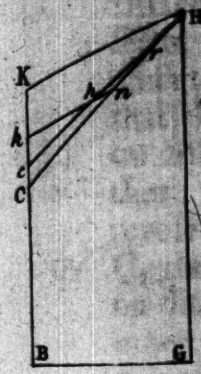


Fig. 73.

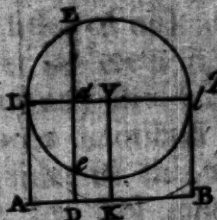
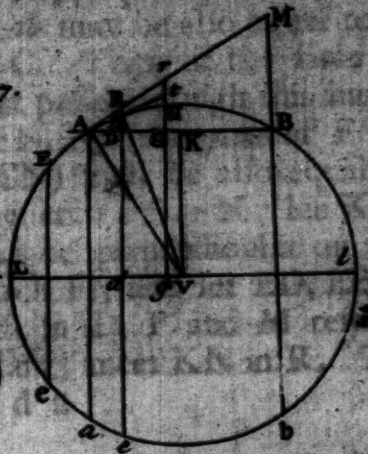


Fig. 77.



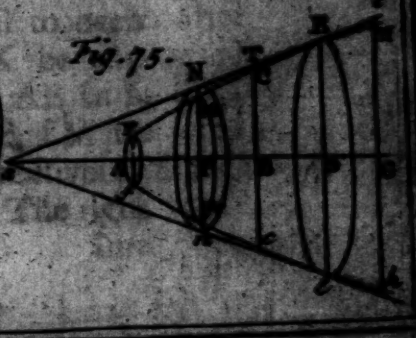


Fig. 75.

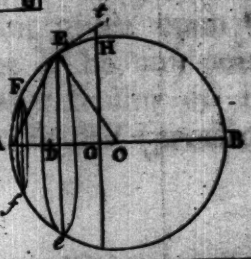
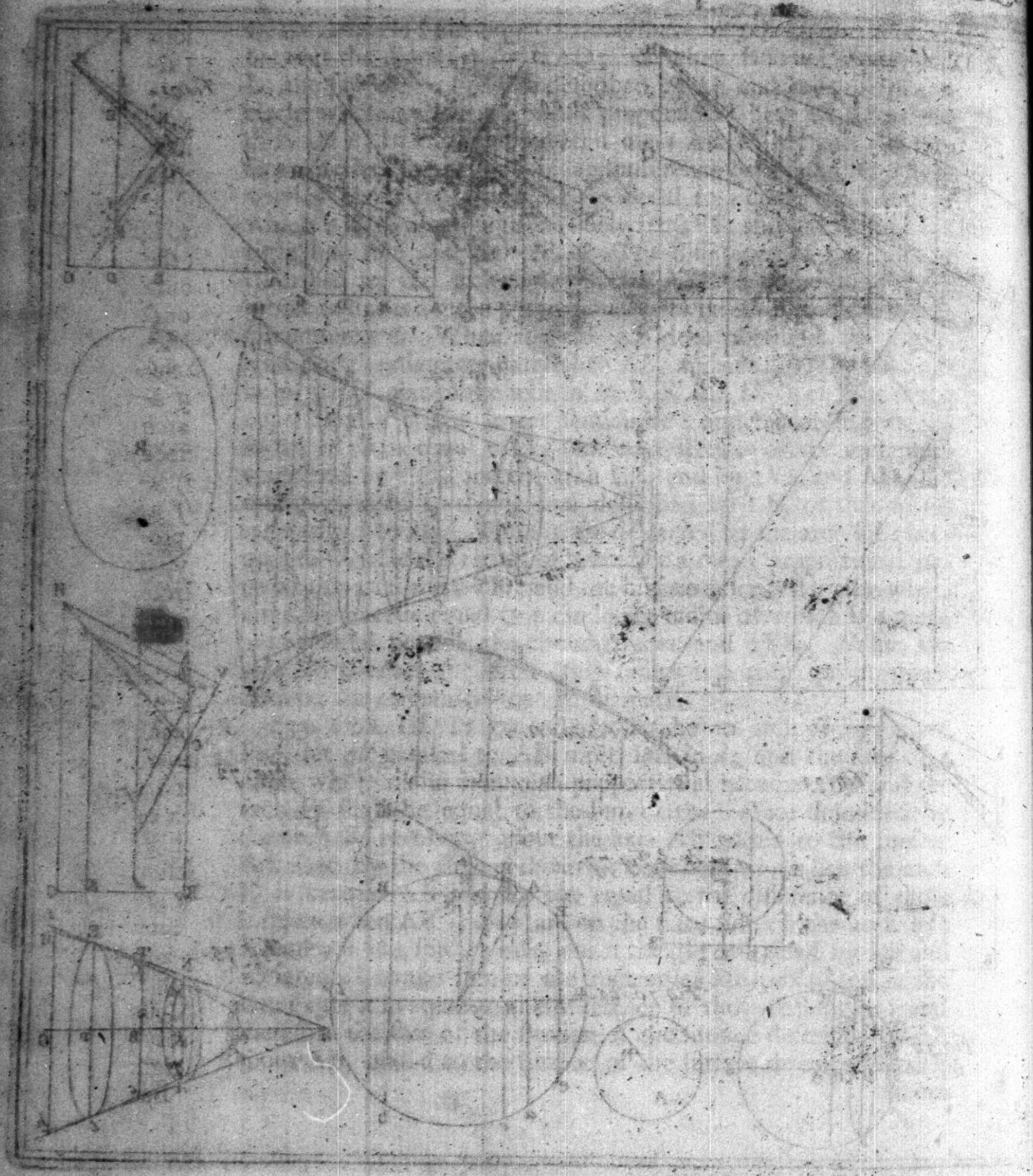


Fig. 76 n. 1.

Fig. 76. n. 2.

R

A



about *ab* in the former caſe, and the difference of thoſe fluxions in the latter caſe, is equal to the fluxion of an area that is always equal to a circle whoſe radius is a mean proportional betwixt 2D*d* and the arch FE: From which it follows, (by theor. 8.) that this circle is equal to the ſum of the ſurfaces deſcribed by FE about AB and *ab* in the firſt caſe, and to their difference in the ſecond caſe.

234. Cor. IV. It follows from the laſt corollary, that, if the axis *ab* meet the arch F*f* in *a*, ſo that the part F*a* be on the ſame ſide of *ab* with the axis AB, and *af* be on the oppoſite ſide of *ab*; then a circle whoſe radius is a mean proportional betwixt 2D*d* and the arch F*f* ſhall be equal to the ſurface deſcribed by F*f* about AB, added to the exceſs of the ſurface deſcribed by F*a* about *ab* above that which is deſcribed by *af* about the ſame axis *ab*. Therefore, if the axis *ab* cut the arch F*f* in ſuch a manner in the point *a*, that the ſurfaces deſcribed by F*a*, *af* (the parts of F*f* that are on oppoſite ſides of *ab*) about the axis *ab* be equal, then the ſurface deſcribed by F*f* about AB ſhall be equal to a circle whoſe radius is a mean proportional betwixt 2D*d* and the arch F*f*. And, converſely, if the ſurface deſcribed by F*f* about AB be equal to ſuch a circle, the ſurfaces deſcribed by the parts of F*f* that are on different ſides of *ab* muſt be equal to each other. Fig. 79.

235. Cor. V. Let AB meet the arch F*f* in A, ſo that the ſurfaces deſcribed by FA, A*f* (the parts of F*f* that are on different ſides of AB) revolving about AB may be equal to each other; let *ab* perpendicular to AB meet the arch F*f* in *a*, ſo that the ſurfaces deſcribed by F*a*, *af* (the parts of F*f* that are on different ſides of *ab*) about *ab* may be alſo equal to each other; and let *ab* meet AB in C. Then, if the ſame arch F*f* revolve about any axis CN that paſſes through this interſection C, the ſurfaces deſcribed by FN, N*f* (the parts of F*f* that are on different ſides of this axis CN) ſhall be alſo equal to each other. For, let CN meet the arch F*f* in N; let K be any point given upon CN, and KI be perpendicular on AB in I. Let E be any point of the arch F*f*, and let ED, EP, EM be perpendicular on KN, *ab*, AB in D, P and M reſpectively; and let EP (produced if neceſſary) meet KN in R. The right Fig. 80.

line ER is the difference of EP and PR when the point E is taken upon the arch FA or *af*, in the caſe repreſented by fig. 80. and ER is the ſum of EP and PR when E is taken upon the arch A*a*. But ED is always to ER as KI is to CK; and, ſince PR is to CP (or EM) as CI is to KI, it follows, that the rectangle contained by CK and DE is equal to the difference or ſum of the rectangle contained by KI and PE and that which is contained by CI and ME; and, the fluxion of the arch AE being repreſented by E*t*, the ſolid contained by CK and the rectangle DE*t* is equal to the difference or ſum of the ſolid contained by KI and the rectangle PE*t* and that which is contained by CI and the rectangle ME*t*, according as E is upon the arches F*a*, *af*, or upon A*a*. Hence the ſolid contained by CK and the ſurface deſcribed by the arch AE (or *a*E) revolving about the axis KN, is equal to the difference of the ſolid that is contained by KI and the ſurface deſcribed by AE (or *a*E) about *ab* and the ſolid contained by CI and the ſurface deſcribed by AE (or *a*E) about AB, when E is taken upon the arch FA or *af*, but to the ſum of thoſe ſolids when E is taken upon the arch A*a*. Therefore the ſolid contained by CK and the ſurface deſcribed by the arch FN about the axis KN is equal to the difference of two ſolids, the firſt of which is that contained by KI and the difference of the ſurfaces deſcribed by F*a* and *a*N about the axis *ab*, the ſecond is the ſolid contained by CI and the difference of the ſurfaces deſcribed by FA and AN about the axis AB. And, in like manner, the ſolid contained by CK and the ſurface deſcribed by the arch N*f* about KN is equal to the exceſs of the ſolid contained by KI and the ſurface deſcribed by N*f* about *ab* above the ſolid contained by CI and the ſurface deſcribed by N*f* about AB. But it follows from the ſuppoſition, that the difference of the ſurfaces deſcribed by the arches F*a* and *a*N about *ab* is equal to the ſurface deſcribed by the arch N*f* about the ſame axis *ab*; and that the difference of the ſurfaces deſcribed by FA and AN about AB is equal to the ſurface deſcribed by N*f* about AB. Therefore the ſurfaces deſcribed by the arches FN, N*f* about the axis KN are equal to each other when the ſurfaces deſcribed by FA and A*f* about AB are equal, and the ſurfaces deſcribed by F*a* and *af* about *ab* are e-

qual.

qual at the ſame time. This demonſtration is eaſily applied to the other caſes that are not repreſented in fig. 80.

236. Cor. VI. It follows from what has been demonſtrated, that, any arch *Ff* being given, there is a certain point C in the plane of this arch, through which any right line KN being drawn meeting the arch in N, the ſurfaces deſcribed by FN, N*f* (the parts of the arch that are on oppoſite ſides of KN) revolving about the axis KN are always equal to each other. This point is called the *center of Gravity* of the arch, (becauſe, if it be ſuppoſed to conſiſt of matter that is acted upon by an uniform gravity in parallel lines, the *momentum* of the part FN about any axis KN that paſſes through C is equal to the *momentum* of the part N*f* about the ſame axis, as we may have occaſion to ſhew afterwards;) and it follows from the two laſt corollaries, that the ſurface deſcribed by the arch *Ff* revolving about *kn*, perpendicular to any line C*c* drawn through C, is equal to the area of a circle whoſe radius is a mean proportional betwixt 2C*c* and the arch *Ff* when *kn* does not cut the arch *Ff*: but the difference of the ſurfaces deſcribed by the parts of the arch *Ff* that are on oppoſite ſides of *kn* is equal to that circle when *kn* cuts the arch *Ff*. The area of ſuch a circle is equal to the rectangle contained by the arch *Ff* and the circumference of the circle deſcribed by the point C when the figure is ſuppoſed to revolve about *kn*; and therefore this corollary agrees with the celebrated theorem commonly aſcribed to Guldinus.

237. Cor. VIII. Hence the diſtance of the point C from any right line *kn* is determined, when the length of the arch *Ff* and the ſurface deſcribed by this arch revolving about *kn* are given. For a third proportional to the length of the arch *Ff* and the radius of a circle that is equal to that ſurface ſhall be double of the diſtance of C from *kn*. Thus, becauſe the ſurface deſcribed by the arch AEB, revolving about the diameter L*l* parallel to the chord AB, is equal to the area of a circle whoſe radius is a mean proportional betwixt the diameter L*l* and chord AB, it follows, that the diſtance of the center of gravity of the arch AEB from the center of the circle is to the radius as the chord AB is to the arch AEB. Fig. 76.

CHAP.

CHAP. IX.

Of the greateſt and leaſt Ordinates, of the points of contrary Flexion and Reflexion of various kinds, and of other affections of Curves that are defined by a common or by a fluxional Equation.

238. THere are hardly any ſpeculations in Geometry more uſeful or more entertaining * than thoſe which relate to the *Maxima* and *Minima*. Several propoſitions of this nature are to be found in the writings of the ancient Geometricians; but they do not ſeem to have had a general method for reſolving problems of this kind. Amongſt the various improvements that began to appear in the higher parts of Geometry about a hundred years ago, Mr. de FERMAT propoſed a method for finding the *maxima* and *minima*. How the methods that were then invented for the menſuration of figures and drawing tangents to curves are comprehended and improved by the method of Fluxions, may be underſtood from what has been already demonſtrated. A general way of reſolving queſtions concerning the *maxima* and *minima* is alſo derived from it, that is ſo eaſy and expeditious in the moſt common caſes, and is ſo ſucceſsful when the queſtion is of a higher degree, where the difficulty is greater, and other methods fail us, that this is juſtly eſteemed one of the moſt admirable applications of Fluxions.

239. When the nature of a variable quantity is ſuch, that it either increaſes continually without end, or decreaſes till it vaniſhes, its greateſt or leaſt magnitude is not aſſignable; and there is no place for enquiries of this nature. But, when there is a certain limit which the increaſe or decreaſe of the variable quantity cannot paſs, and the term is aſſignable when it arrives at this limit; or, more generally, when for ſome time the variable quantity firſt increaſes till a certain aſſignable term, and then

* Apoll. Perg. Conic. lib. 5. præf.

decreaſes,

decreaſes, or firſt decreaſes till ſuch a term and then increaſes: its magnitude at that term is conſidered as a *maximum*, or *minimum*, without regard to its variations in other parts of the time.

240. In the problems of this kind of the firſt degree, the variable quantity is repreſented by an ordinate of a curve the nature of which is ſuppoſed to be defined by what is given concerning the variable quantity. A curve line either returns into itſelf, or may be continued without end; and therefore there are always two branches of the curve that proceed from any point that is aſſignable in it. The ordinate from a point of the curve is a *maximum*, or *minimum*, when it is greater or leſs than the ordinates which may be drawn from the parts of either branch of the curve adjoining to that point. When the curve is continued immediately from that point on both ſides of the ordinate, we ſhall call the ordinate a *maximum* or *minimum* of the *firſt kind*, but of the *ſecond kind* when the curve is reflected from that ordinate and both the branches of the curve are on the ſame ſide of it. Sometimes three or more branches of a curve proceed from the ſame point; but the ordinate in that caſe is only to be compared with the ordinates from theſe two branches that are to be conſidered, according to the nature of the curve, as the immmediate continuation of each other. FIG. 81. & 82. FIG. 83.

241. The writers on this ſubject do not always agree as to the extent of the problems they comprehend under this claſs. Some * have propoſed to comprehend under the *maxima* and *minima* the greateſt or leaſt ordinates that can be drawn from the part of the curve that is convex towards the baſe, or from that which is concave towards the baſe, though the ordinates from the adjoining parts of the ſucceeding branch of the curve may be greater or leſs than that ordinate. Others, who exclude that caſe, ſeem alſo to exclude the greateſt or leaſt ordinate when it paſſes through a point where the curvature is not continued, that is, a point of *Reflexion*, or *Cuſpis*. Others comprehend the ordinate from a cuſpis when the two branches of the curve that proceed from it are on different ſides of the ordinate, but exclude ſuch an ordinate when the two branches of

* Deſcartes's letters, tom. 3. let. 60.

the

the curve that proceed from the cuſpis are on the ſame ſide of the ordinate, becauſe thoſe branches (or at leaſt the adjoining parts of each) are over the ſame baſe. But it ſeems to be more conſiſtent, to include all ordinates that are greater, or leſs, than thoſe from the adjoining parts of either branch of the curve; the rather, that in the latter caſe one of the branches of the curve, after being reflected from the ordinate, often returns to it again, and after cutting it, proceeds on the other ſide, ſo that one or more ordinates may correſpond to any aſſignable baſe. However, ſince it has been more uſual to exclude this caſe, we have diſtinguiſhed the greateſt and leaſt ordinates into two kinds in the preceeding article, to prevent miſtakes.

FIG. 83.

FIG. 81.

242. When an arch of a curve has its concavity turned one way, and there is a point in this arch where the tangent becomes parallel to the baſe, the greateſt or leaſt ordinate paſſes through that point. It is greater or leſs than thoſe from the adjoining parts of the arch on either ſide, according as the arch is concave or convex towards the baſe. Suppoſing the baſe to increaſe, the ordinate in the former caſe firſt increaſes and then decreaſes, and in the latter caſe the ordinate firſt decreaſes and then increaſes. In both caſes, the motion with which the baſe flows (or its fluxion) being given, the motion with which the ordinate flows (or its fluxion) firſt decreaſes and then increaſes, by the 7th lemma: And this motion, or the fluxion of the ordinate, vaniſhes when the tangent becomes parallel to the baſe. For, EI (fig. 47. & 50.) being ſuppoſed to meaſure the fluxion of the baſe, IT which meaſures the fluxion of the ordinate (by prop. 14.) vaniſhes when the tangent ET becomes parallel to the baſe and coincides with EI. Therefore the greateſt and leaſt ordinates are diſcovered in ſuch caſes, (which are thoſe that moſt commonly occur in the reſolution of problems,) by enquiring when the fluxion of the curve becomes equal to the fluxion of the baſe, or when the fluxion of the ordinate vaniſhes the fluxion of the baſe being given. In this caſe however the curve muſt be continued on both ſides of the ordinate: for, if the curve be reflected from the ordinate DE, then the point E is a cuſpis, and the ordinate DE is neither a *maximum* nor *minimum* when the arches CE, EH have their convexity towards

wards each other; and it is a *maximum* or *minimum* of the ſe- Fig. 84.
cond kind when thoſe branches are on the ſame ſide of the tan- Fig. 85.
gent at E, the convexity of one branch being towards the con-
cavity of the other.

243. The arch being ſuppoſed to have its concavity turned
one way, and the tangent at E being ſuppoſed parallel to the Fig. 81.
baſe, if the arch meet the baſe we may conclude that DE is a n. 1.
maximum; but if the arch be of that kind which may be con- Fig. 81.
tinued above the baſe without end, DE is a *minimum*. The n. 2.
greateſt and leaſt ordinates are diſtinguiſhed from each other
more generally, by comparing them with the ordinates from
the adjoining parts of the curve; or by ſuppoſing the baſe AP
to increaſe, and obſerving whether the fluxion of the ordi-
nate PM is poſitive before the point P comes to D and becomes
negative after P paſſes D, or is firſt negative and then becomes
poſitive, the ordinate being itſelf conſidered as poſitive: It is
a *maximum* in the former, and a *minimum* in the latter caſe.

244. In the application of this rule for finding the greateſt
and leaſt ordinates, it muſt be obſerved, that the ratio of the
fluxion of the ordinate to the fluxion of the baſe may be ſome-
times repreſented by the ratio of two quantities which at cer-
tain terms may both vaniſh together: but it does not follow,
that in ſuch caſes the tangent becomes parallel to the baſe, or
that the ordinate is a *maximum* or *minimum*; for the ratio of
thoſe fluxions may be repreſented in ſuch caſes by that of other
finite and aſſignable quantities. When two or more branches
of the curve are over the ſame part of the baſe, the ordinate muſt
ariſe to two or more dimenſions in the equation of the curve.
The fluxion of the baſe being given, the fluxion of the ordinate
is at leaſt twofold; and when the ordinate paſſes through a point
where two or more branches of the curve interſect each other,
we are to expect, that though the baſe and ordinate be both gi-
ven, the fluxion of the ordinate will ariſe to two or more di-
menſions in the general equation by which it is to be determi-
ned. But if the general equation of the curve can be reſolved
into theſe particular equations which belong to the different
branches of the curve from which the general equation is com-
pounded, then the fluxion of each ordinate may be determined

from the fluxion of the equation of each branch. But we may have occaſion to illuſtrate this further afterwards.

FIG. 82. 245. When the fluxion of the curve coincides with the fluxion of the ordinate, the tangent becomes perpendicular to the baſe, and coincides with the ordinate. In this caſe, if EC, EH the two branches of the curve form a cuſpis at E, being on different ſides of the ordinate DE, then is DE a *maximum* or *minimum* according as thoſe branches have their convexity or concavity towards the baſe. The *maxima* and *minima* of this ſort are diſcovered, by enquiring when the fluxion of the curve becomes equal to the fluxion of the ordinate, or (which is the ſame thing) when the fluxion of the baſe vaniſhes the fluxion of the ordinate being given. In this caſe alſo it is neceſſary, that the curve be continued immediately from E on both ſides of the ordinate DE; for if the two branches EC, EH be on the ſame ſide of DE, the ordinate is either no *maximum* nor *minimum*, or is one of the ſecond kind.

246. Theſe are the two rules that are commonly given for determining the *maxima* and *minima* of the firſt kind. But there is ſtill another limitation beſides thoſe already mentioned, without which theſe rules may lead us into error. For we are not always to conclude that the ordinate DE is a *maximum* or *minimum*, either when the fluxion of the ordinate vaniſhes the fluxion of the baſe being given, or when the fluxion of the baſe vaniſhes the fluxion of the ordinate being given, though the curve be continued immediately on both ſides of the ordinate: In the former caſe the tangent at E is parallel, in the latter per-

FIG. 91. pendicular to the baſe; but E may be a point of contrary flexion, ſo that the ordinates on one ſide of DE may be greater than DE, and thoſe on the other ſide leſs than it; and there may be no *maximum* nor *minimum* perhaps from the whole curve. In any curve that has a point of contrary flexion, the ordinate diſcovered by thoſe rules is not a *maximum* or *minimum* if the baſe be parallel or perpendicular to the tangent at that point. The rules which are given for finding the points of contrary flexure are liable to exceptions of the ſame nature for a ſimilar reaſon. In order to ſet theſe rules and the neceſſary exceptions in a clear light, it will be of uſe to premiſe the following propoſition.

PROP.

PROP. XX.

247. *Let the ordinate* DE *of the curve* AEH *meet the curve* Feh *in* e, *and the rectangle* EG *contained by* DE *and a given right line* DG *be always equal to the area* ADeF; *then, the rectangle* DeLG *being completed,* ET *the tangent of the curve* AEH *at* E *ſhall be parallel to the diagonal* DL. *And, converſely,* EK *being equal and parallel to* DG, *if* KT *parallel to* DE *meet the tangent* ET *in* E, *and,* De *being taken always equal to* KT, *if the curve* HEC *meet the baſe in* A, *the area* ADeF *ſhall be equal to the rectangle* EG. Fig 86.

The fluxion of the baſe AD being repreſented by the given right line DG or EK, the fluxion of the ordinate DE is repreſented by KT, by prop. 14. and the fluxion of the area ADeF by the rectangle eG, by prop. 3. The rectangle EG is always equal to the area ADeF by the ſuppoſition; and the fluxion of EG is equal to the fluxion of the area ADeF, by art. 18. that is, the rectangle contained by KT and DG is equal to eG, and KT is equal to De or GL. Therefore the tangent ET is parallel to DL.

248. And, converſely, the right line DG or EK being given, if upon the ordinate DE a right line De be taken always equal to KT, the rectangle eG ſhall be always equal to the rectangle contained by KT and DG; that is, the fluxion of the area ADeF ſhall be always equal to the fluxion of the rectangle EG; and, by theor. 4. the fluents generated in the ſame time being equal, it follows, that ſince the curve CEH is ſuppoſed to paſs through A, and the point E ſets out from A when e ſets out from F and the right line De from AF, the area ADeF muſt be equal to the rectangle EG. The analogy there is betwixt the inverſe method of tangents and the quadrature of curvilineal figures appears from this propoſition.

249. It may be of uſe for illuſtrating this doctrine, to demon-

ſtrate in the following manner, (which is independent of the method of fluxions,) that if the rectangle EG be always equal to the area ADeF, then ET parallel to DL ſhall be the tangent of the arch EH at E. For, let any ordinate PM meet the arches EH, *eh*, the right lines EK, eL parallel to the baſe, and ET parallel to DL, in the points M, N, V, S and R, reſpectively. Then, ſince the rectangle contained by VR and DG is to the rectangle contained by KT and DG as VR is to KT, or DP to DG, or as the rectangle *e*P is to *e*G; and the rectangle contained by KT and DG is equal to *e*G; it follows, that the rectangle contained by VR and DG is equal to the rectangle *e*P. But by the ſuppoſition the rectangle contained by PM and DG is equal to the area APNF; and the rectangle contained by DE and DG is equal to the area ADeF: therefore the rectangle contained by VM and DG is equal to the area DPN*e*; and VM is to VR as the area DPN*e* is to the rectangle *e*P. But when the ordinates from the arch *eh* increaſe while the baſe increaſes, the area DPN*e* always exceeds the rectangle *e*P. Therefore in this caſe VM is always greater than VR, and the arch EH is all above the right line ET, the point E only excepted. Nor can any right line be drawn through E within the angle of contact HET. For, ſince KH is to KT (or GL) as the area DG*he* is to the rectangle *e*G, and therefore in a leſs ratio than G*h* is to GL, it follows, that KH is leſs than G*h*. Therefore, Q being any point betwixt T and H, if DO be taken equal to KQ, it ſhall be leſs than G*h*, and a right line through O parallel to the baſe ſhall meet the arch *eh* in ſome point betwixt *e* and *h*. Let this point be N, and let the ordinate PN meet the arch EH and the right lines EQ, EK, *e*L in the points M, Z, K and S. Then, ſince the rectangle contained by KQ and DG is to the rectangle contained by VZ and DG as KQ is to VZ, or DG to DP, or as the rectangle OG is to OP, it follows, that the rectangle contained by VZ and DG is equal to OP, and that VZ is to VM as the rectangle OP is to the area DPN*e*. Therefore VZ is greater than VM, and the right line EQ does not paſs through the angle of contact HET, but cuts the arch EH in ſome point betwixt M and H. From which it follows, by art. 181. that ET is the tangent of the arch EH at E.

250. In the ſame manner, if any ordinate *pn* meet the arches *ce*, CE and the right lines E*t*, E*k*, *el* (which are the right lines TE, KE, L*e* continued beyond E and *e*) in *n*, *m*, *r*, *v* and *f*, *vm* ſhall be to *vr* as the area D*pne* is to the rectangle *ep*; and therefore, when the ordinates from *ce* increaſe while the baſe increaſes, the arch EC is all above E*t*, the point E excepted. In the ſame manner, *k*C is greater than B*c*, (the rectangle contained by *k*C and DG being equal to the area DB*ce* which exceeds the rectangle contained by B*c* and DG;) and *kt* being equal to KT or D*e*, if *q* be any point betwixt C and *t*, and D*o* be taken equal to *kq*, a parallel to the baſe through *o* ſhall meet the arch *ce* in ſome point betwixt *c* and *e*. Let that point be *n*, and let the ordinate *pn* meet E*q* in *z*; then, ſince *kq* (or D*o*) is to *vz* as DB is to D*p*, or as the rectangle *o*B is to *op*, it follows, that the rectangle contained by *vz* and DG is equal to the rectangle *op*, and that *vz* is to *vm* as *op* is to the area D*pne*. Therefore *vz* is leſs than *vm*; and, ſince *k*C is leſs than *k*Q, the right line E*q* does not paſs through the angle of contact CE*t*, but interſects the arch EH in ſome point betwixt *m* and C. Therefore E*t*, which is the right line ET continued, is the tangent of the arch EC which is the continuation of the arch HE. Becauſe the tangent *t*T is betwixt the curve CEH and the baſe in this caſe, it appears, that when the ordinates from the arch *ceh* increaſe, (the baſe AP being ſuppoſed to increaſe,) the arch CEH is convex towards the baſe.

251. When the ordinates from the arch *ceh* decreaſe while the baſe increaſes, the rectangle *e*P exceeds the area DPN*e*, and the rectangle *ep* is leſs than the area D*pne*. Therefore VR is greater than VM, and *vr* leſs than *vm*; ſo that the whole arch HEC is below the right line TE*t*, the point E excepted: And it is ſhewn in the ſame manner as in the preceeding caſe, that no right line can be drawn through the angles of contact HET, CE*t*. Therefore, in this caſe, the right line T*t* is the tangent at E, and the arch CEH is concave towards the baſe. In both caſes the curve CEH paſſes through A, becauſe when AP vaniſhes, the rectangle contained by PM and DG (which is ſuppoſed to be always equal to the area APNF) vaniſhes; and if A*h* be taken towards G equal to DG, and the rectangle A*hf*F be- FIG. 87.

be completed, the diagonal *Af* ſhall be the tangent at A.

Fig. 86. 252. Cor. I. Let *Zez* the tangent of the arch *ceh* at *e* meet *Gh* and *Bc* in Z and *z*, and let the ordinates from *Bc* to *Gh* increaſe continually; then, if the arch *ceh* be convex towards the baſe, TH which ſubtends the angle of contact HET ſhall be leſs than one half of *Lh*, but greater than one half of LZ. For the rectangle contained by TH and DG is equal to the area *eLh* which is leſs than the triangle *eLh*, or one half of the rectangle contained by DG and *Lh*, but is greater than the triangle *eLZ*, or one half of the rectangle contained by DG and LZ. Therefore TH is leſs than one half of *Lh*, but greater than one half of LZ. In the ſame manner it appears, that *t*C is leſs than one half of *lz*, but greater than one half of *lc*.

Fig. 87. 253. Cor. II. When the arch *ceh* is concave towards the baſe, TH is leſs than one half of LZ, but greater than one half of *Lh*. For, in this caſe, the area *eLh* is leſs than the triangle *eLZ*, but greater than the triangle *eLh*. In like manner *t*C is in this caſe leſs than one half of *lc*, but greater than one half of *lz*.

254. Cor. III. When *ceh* is a right line and coincides with *zeZ*, CEH is an arch of a parabola that has its axis perpendicular to the baſe AG. In this caſe TH and *t*C are each equal to one half of LZ which meaſures the fluxion of D*e* (by prop. 14.) or the ſecond fluxion of DE, the fluxion of the baſe being repreſented by DG. While the baſe AD acquires the augment DG, the ordinate DE acquires the augment KH equal to the ſum of KT and TH; and in this caſe the firſt fluxion of the ordinate is repreſented by KT, and its ſecond fluxion by 2TH, or the ſum of TH and *t*C. But when *ceh* is convex towards the baſe, and, the baſe being ſuppoſed to flow uniformly, its fluxion is repreſented by DG, the right line TH is greater than one half of LZ which meaſures the ſecond fluxion of the ordinate DE, but leſs than one half of *Lh* which meaſures the increaſe of the fluxion of the ordinate that is generated in the ſame time in which the baſe acquires the augment DG. When the arch *ceh* is concave towards the baſe, TH (which ſubtends the angle of contact HET) is leſs than one half of the right line that meaſures the ſecond fluxion of DE, but greater than one half of the right line that meaſures the increaſe of the fluxion

xion of DE. In the firſt caſe, when *ceh* is a right line, the motion with which DE flows is uniformly accelerated. When *ceh* is convex towards the baſe, the acceleration of that motion increaſes; and when *ceh* is concave towards the baſe, its acceleration decreaſes continually.

255. Cor. IV. When the curve A*ch* is a parabola that has its axis perpendicular to the baſe AG, KH the increment of DE may be diſtinguiſhed into three parts, KT, TQ and QH, ſo that the rectangle contained by thoſe parts and DG may be reſpectively equal to the rectangle *e*G, the triangle *e*LZ, and the area *eZh*. The part KT is equal to D*e*, and meaſures the firſt fluxion of DE; the part TQ is equal to one half of LZ, which meaſures the ſecond fluxion of DE, (by cor. 3.) and the part QH is equal to one third part of *Zh*, (that meaſures one half of the fluxion of LZ,) and therefore meaſures one ſixth part of the third fluxion of DE. For it follows from what was ſhewn in the Introduction (pag. 27.) after Archimedes, that the area *eZh* is one third part of the rectangle contained by *Zh* and DG: And it may be eaſily deduced from the 8th propoſition; for let PM meet the arch *eh* in N, and its tangent *e*Z in *u*; and *u*N ſhall be to *Zh* as the ſquare of DP is to the ſquare of DG. Therefore, the point D and the right lines DG, *Zh* being given, but ſuppoſing DP to flow, and DG, DP, X and Y to be in continued proportion, it will follow from the eighth propoſition, that one third part of the fluxion of Y ſhall be to the fluxion of DP as Y is to DP, or as the ſquare of DP is to the ſquare of DG, and therefore as *u*N is to *Zh*; ſo that the rectangle contained by *u*N and the right line which meaſures the fluxion of DP is equal to one third part of the rectangle contained by *Zh* and the right line which meaſures the fluxion of Y. Therefore the area *e*UN is equal to one third part of the rectangle contained by *Zh* and Y, or (becauſe *Zh* is to *u*N as DP is to Y) of the rectangle contained by *u*N and DP; and the area *eZh* is equal to one third part of the rectangle contained by *Zh* and DG. In the ſame manner it is ſhewn, that the area ERM is equal to one fourth part of the rectangle contained by RM and DP; and the continuation of theſe theorems is obvious from the ſame eighth propoſition. From which it follows, Fig. 88.

lows, that when the fluxions of all orders of the ordinate DE increaſe, we approximate continually to the value of KH, the increment of the ordinate that is generated in the ſame time the baſe acquires the augment DG, by adding continually together the right line that meaſures the firſt fluxion of DE while DG meaſures the fluxion of the baſe, $\frac{1}{2}$ of that which meaſures the ſecond fluxion of the ordinate, $\frac{1}{6}$ of that which meaſures its third fluxion, $\frac{1}{24}$ of that which meaſures its fourth fluxion, and ſo on, the denominators of thoſe fractions being the products of the numbers 1, 2, 3, 4, 5, *&c.* in their natural order. But when any fluxion decreaſes, the ſucceeding fluxion is to be conſidered as negative, and the fraction which involves it is to be ſubducted. Theſe corollaries illuſtrate what was ſhewn of ſecond and third fluxions near the end of the firſt and fourth chapters.

FIG. 89. 256. Let any point E be given in the curve CEH; let KE*k* be a right line parallel to the baſe AD, PM an ordinate meeting EK in V and the curve *ceh* in N, and the rectangle contained by VM and the given right line DG be always equal to the area DPN*e*. Suppoſe the points P and M to move from D and E; and, when AP increaſes and PN is above the baſe AP, let VM be taken upon PV produced beyond V: then, when AP decreaſes, if PN be ſtill above the baſe, or when AP increaſes if PN be below the baſe, VM is to be taken upon VP from V towards P; but if AP decreaſe and PN be below the baſe, VM is to be taken upon PV produced beyond V.

FIG. 90. 257. When the point *e* in the curve *Fch* falls on the baſe and coincides with D, DL coincides with DG, the tangent *t*ET coincides with *k*EK and becomes parallel to the baſe. In this caſe, if the curve *Fch*, after meeting the baſe AD, be continued on the other ſide of AD and on the other ſide of the perpendicular DE, then is DE a *maximum*; for, by the laſt article, P being taken on either ſide of D, VM is to be taken from V towards P. When PM meets the curve *Fch* below the baſe in N and the curve CEH above it in M, the rectangle contained by VM and DG being equal to the area DPN, it follows, that the rectangle contained by PM and DG is equal to the exceſs of the area ADF above the area DPN, and that

PM

PM vaniſhes when thoſe areas become equal. If the curve FDN return towards the baſe, and after cutting it again in d be continued on the oppoſite ſide of the baſe and of the perpendicular de, then de the ordinate of the curve AEM at d is a *maximum* or *minimum* according as it meets the curve above or below the baſe. In the ſame manner, if the curve NF continued beyond F meet the baſe in *a*, and proceed from *a* below the baſe on the other ſide of the perpendicular at *a*, the curve MEA ſhall be continued from A below the baſe, and its ordinate at *a* ſhall be a *maximum*.

258. By the 256th article, the form of the arch *ceb* being given, we diſcover the form of the arch CEH. Suppoſe now the form of the arch CEH to be known, and that of *ceb* to be required. Let E be any point in the arch CEH, and ſuppoſe that D*e* is taken above the baſe when KT is upon GK produced beyond K. Then, if the tangent ET meet GK in T betwixt G and K, or B*k* produced beyond *k* in *t*, the ordinate D*e* is to be taken upon ED produced beyond D below the baſe; but when ET meets B*k* in *t* betwixt B and *k*, D*e* is to be taken above the baſe. Fig. 89.

259. From this the converſe of the 257th article is manifeſt: That when the ordinate DE is a *maximum* or *minimum* of the firſt kind, and the tangent at E is parallel to the baſe, the arches *ec*, *eb* of the curve *ceb* muſt be continued from *e* (which coincides with D in this caſe) on different ſides of the baſe AD and of the perpendicular at D. For, while the point M deſcribes the arches CE, EH, the ordinate PN of the curve *ceb* muſt be taken on different ſides of the baſe. Fig. 90.

260. But when the curve FND, after meeting the baſe in D, is continued on the oppoſite ſide of DE, but on the ſame ſide of the baſe as before, then the ordinate DE is not a *maximum* or *minimum*, though PN which meaſures the fluxion of the ordinate vaniſhes when P comes to D. For, while N deſcribes the archs *c*D and D*b*, the right line VM is to be taken on different ſides of *k*K, which is parallel to the baſe. Therefore DE is not a *maximum* or *minimum*, but E is a point of contrary flexure, whether the arch *c*D*b* touch the baſe in D, or form a cuſpis there. It is manifeſt, converſely, from art. 258. that, when E is Fig. 91.

is a point of contrary flexure and the tangent at E is parallel to the baſe, the arch *cb*, after meeting the baſe in D, is continued on the ſame ſide of the baſe; becauſe the ordinates from BC to GH increaſe or decreaſe continually: and, MI being equal and parallel to DG, if IT parallel to DE meet the tangent at M in t, the right line It is on the ſame ſide of MI while M deſcribes the arches CE and EH.

PROP. XXI.

261. *The fluxion of the baſe being given, and the curve being continued on both ſides of the ordinate, when the firſt fluxion of the ordinate and its fluxions of any number of ſubſequent ſucceſſive orders vaniſh, the ordinate is a* maximum *or* minimum, *or paſſes through a point of contrary flexion, according as that number is even or odd.*

FIG. 90. & 91. The curve being continued on both ſides of the ordinate, and the fluxion of the baſe being given, let the firſt fluxion of the ordinate DE vaniſh; then, if the number of its ſubſequent ſucceſſive fluxions that vaniſh be 0, 2, 4, or any even number, the ordinate DE ſhall be a *maximum* or *minimum*; but if that number be 1, 3, 5, or any odd number, E ſhall be a point of contrary flexure. Firſt, let the curve *cb* cut the baſe in D in any aſſignable angle, and be continued from D on oppoſite ſides of the baſe and ordinate. In this caſe DE is a *maximum* or *minimum* by art. 257. and the firſt fluxion of DE vaniſhes, but the ſecond fluxion does not vaniſh. In the next place, let the arch GH be ſubſtituted in place of *cb*, and a new curve be derived from CH in the ſame manner that CH was derived from *cb*: Then DE ſhall meet this third curve in a point of contrary flexure, by art. 260. and the firſt and ſecond fluxions of the ordinate of this curve vaniſh. If this third curve be ſubſtituted for *cb*, and a fourth be derived from it in the ſame manner, the ordinate of this fourth curve at D ſhall be a *maximum* or *minimum*, and its firſt, ſecond and third fluxions, vaniſh.

Pl. X. Pa. 226.

Fig. 78. N. 2.

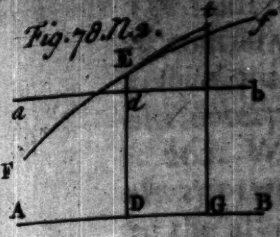


Fig. 78. N. 1.

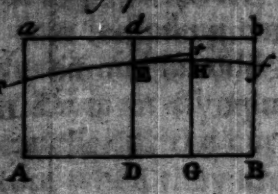


Fig. 80.

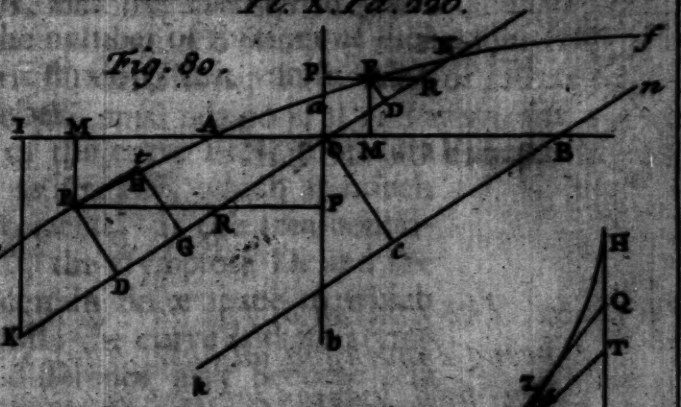


Fig. 81. N. 1.

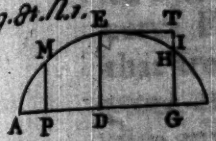


Fig. 79.

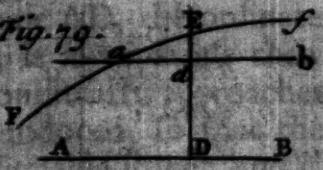


Fig. 81. N. 2.

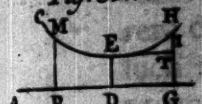


Fig. 83.

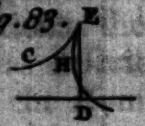


Fig. 84.

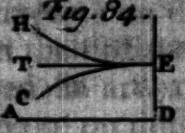


Fig. 85.

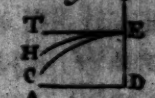
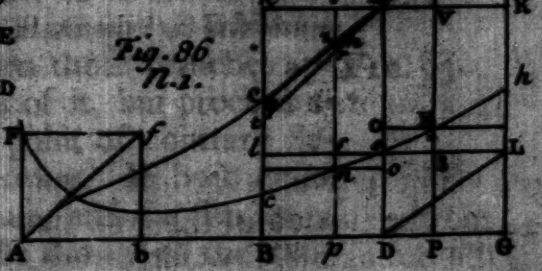


Fig. 86 N. 1.



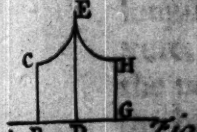


Fig. 82.

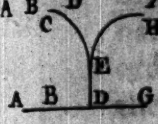


Fig. 88.

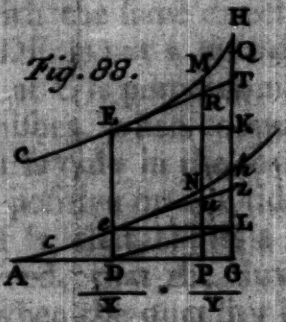


Fig. 87.

Fig. 86. N. 2.

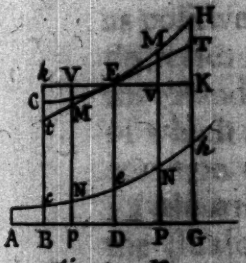


Fig. 87. N. 2.

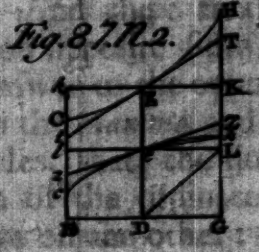


Fig. 89. N. 1 & 2

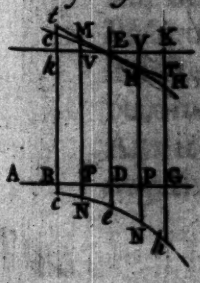


Fig. 90.

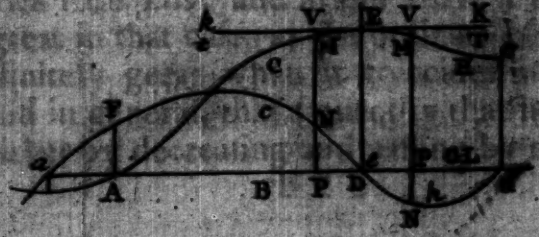


Fig. 91.

niſh. This ſeries of curves being continued, it is manifeſt that DE ſhall be a *maximum* or *minimum*, and paſs through a point of contrary flexure, alternately; the number of fluxions of ſucceſſive orders (not including the firſt fluxion) that vaniſh being alternately an even and an odd number.

262. When *De* is an aſymptote of the curve F*c*N, there are always two branches of the curve, as *c*N, *b*N, which approach to the aſymptote when they are produced. If the area bounded by the baſe AD, the ordinate AF, the aſymptote *De* and the curve F*c*N, is always leſs than a certain finite ſpace to which that area continually approaches while the curve and its aſymptote are produced, ſo that their difference may become leſs than any aſſignable ſpace; and if the two branches *c*N, *b*N proceed alongſt the aſymptote on different ſides of it and in different directions, then the ordinate DE is a *maximum* or *minimum* ſuch as was deſcribed in art. 245. When thoſe branches approach to the aſymptote on different ſides of it, but proceed alongſt it with the ſame direction, E is a point of contrary flexure, and DE is not a *maximum* or *minimum*. In both caſes, the tangent at E is perpendicular to the baſe, and the fluxion of the baſe vaniſhes, when compared with the fluxion of the ordinate; but it is only in the former caſe, when the fluxion of the ordinate is poſitive on one ſide of DE and negative on the other ſide of it, that DE is a *maximum* or *minimum*. We have often obſerved, that the fluxion of a quantity is conſidered as poſitive when the quantity increaſes, but as negative when it decreaſes: And theſe fluxions are repreſented by right lines that ſtand on oppoſite ſides of the baſe, the contrary poſitions of which anſwer to the oppoſite affections of the fluxions. It is common to the *maxima* and *minima* deſcribed in the 242d and 245th articles, that, the fluxion of the baſe AP being poſitive, the fluxion of the ordinate PM is poſitive on one ſide of DE and negative on the other: But in the former it vaniſhes when PM coincides with DE; and in the latter it is ſaid to become infinitely great in that caſe, as the ordinate of a curve is ſaid to become infinitely great when it coincides with an aſymptote. And it is ſaid in general, that a quantity that is poſitive becomes negative, either by decreaſing and paſſing by nothing, or by increaſing

Fig. 92.

Fig. 93.

creasing and passing by infinity: As, if we suppose a right line FQ revolving about a given point F always to intersect the right line Oo in P, and FA to be perpendicular to Oo in A, the right line AP, being first considered as positive, becomes negative by decreasing and passing by nothing, when FQ being supposed to move from FO towards Fo, the intersection P passes through A to the other side of A; but AP is said to become negative by increasing and passing by infinity, when FQ moves in a contrary direction, and, after becoming parallel to Oo, meets it again on the other side of A. It is usual to explain analogies betwixt figures in the common doctrine of curve lines in this manner, and it is often of use in transferring readily the properties of one curve to another that is under the same *genus*. Thus authors explain how the ellipse is transformed into a parabola, and thence into an hyperbola. But, lest conceptions or expressions of this kind should be excepted against, we have endeavoured to avoid them.

Fig. 94.

PROP. XXII.

263. *The ordinate meets the curve in a point of contrary flexure when its fluxion is a* maximum *or* minimum, *the fluxion of the base being given and the curve being continued on both sides of the ordinate.*

Resuming the construction of the 249th and 251st articles, it follows from what was demonstrated there, that, when the ordinates of the arch *ce* increase from Be to De, the arch CE is convex towards the base, and that when the ordinates from De to Gh decrease, the arch EH is concave towards the base; that is, when De is a *maximum*, and the arches *ce*, *eh* are on different sides of De, the point E is a point of contrary flexure. But De represents the fluxion of the ordinate DE, the fluxion of the base being represented by the given right line DG. Therefore, when the fluxion of the ordinate is a *maximum*, and the curve is continued from the ordinate on both sides, it meets the curve in a point of contrary flexure. In like manner, when the ordinates from the arch *ce* decrease and those from *eh* increase, (that

Fig. 95.

Fig. 96.

(that is, when De is a *minimum*,) the arch CE is concave and the arch EH is convex towards the base, by what was shewn in the 249th and 251st articles. Therefore, when the fluxion of the ordinate is a *minimum*, the fluxion of the base being given and the curve being continued from the ordinate on both sides, the ordinate meets the curve in a point of contrary flexure.

264. The proposition appears also from the converse of the 7th lemma, art. 184. For, if we suppose the fluxion of the ordinate DE to be a *maximum*, the fluxion of the ordinate PM must increase while M describes CE, and decrease while M describes EH. Therefore, by the converse of the 7th lemma, if PM increases while M describes CEH, the arch CE must be convex and EH concave towards the base; so that E must be a point of contrary flexure. If PM decrease while M describes CEH, the arch CE must be concave and the arch EH convex towards the base, and E a point of contrary flexure. In the same manner, when the fluxion of DE is a *minimum*, it appears that E is a point of the same kind. We do not comprehend under the *maxima* or *minima* quantities that vanish, or such as are supposed to exceed all assignable magnitude.

Fig 95. Fig. 96.

265. Cor. I. As there are various kinds of *maxima* and *minima*, so there are various kinds of points of contrary flexure. As in the most common cases, the ordinate is a *maximum* or *minimum* when its fluxion vanishes, the fluxion of the base being given; so, when the second fluxion of the ordinate vanishes, the ordinate most commonly passes through a point of contrary flexure. But this is not universally true though the curve be continued on both sides of that ordinate. For, when the tangent of the curve *ceb* at *e* becomes parallel to the base, the second fluxion of DE, or the first fluxion of *De*, vanishes, the fluxion of the base being given; but in this case *e* may be a point of contrary flexure, (art. 260.) and, *De* (which measures the fluxion of DE) not being a *maximum* or *minimum*, the point E is not a point of contrary flexure, but the whole arch CEH has its concavity turned the same way.

Fig. 97.

266. Cor. II. In general, whether the first fluxion of the ordinate vanish or not, the point E is a point of contrary flexure when the number of the subsequent successive orders of fluxions

Fig. 95. 96. & 97.

xions of DE that vaniſh is an odd number, if the curve be continued from the ordinate on both ſides; but E is not a point of contrary flexure, and the whole arch CEH has its concavity turned the ſame way when that number is even; for in the former caſe De is a *maximum* or *minimum*, and in the latter caſe *e* is a point of contrary flexure, by art. 261.

267. When De or the fluxion of the ordinate DE is ſuch a *maximum* or *minimum* as was deſcribed in the 245th and 262d articles, E is a point of contrary flexure; and, the tangent at *e* being perpendicular to the baſe, the right line that meaſures the fluxion of the baſe vaniſhes in this caſe when that which meaſures the fluxion of De, or the ſecond fluxion of DE, is given. But it does not follow, converſely, that when this happens, E is always a point of contrary flexure, although the curve be continued on both ſides of the ordinate DE. For, if *e* itſelf be
Fig. 98. a point of contrary flexure in the curve FN, then, though the tangent at *e* be perpendicular to the baſe, De is not a *maximum* or *minimum*, and E in the curve CEH is not a point of contrary flexure; but the whole arch CH has its convexity or concavity towards the baſe, according as the ordinates of the arch *ch* continually increaſe or decreaſe from *Be* to *Gh*.

268. Hitherto we have ſuppoſed the two arches CE and EH to be on different ſides of DE. When theſe arches are on the ſame ſide of DE and have a tangent at E different from DE, then
Fig. 99. E is a point of reflexion, or cuſpis. The celebrated author of
& 101. the *Analyſe des infiniment petits* diſtinguiſhes thoſe points into two kinds; the point E is a cuſpis of the firſt kind when the arches EC, EH have their convexity towards each other, but of the ſecond kind when the convexity of the one is towards the concavity of the other.

269. Let the arch *ch* meet the baſe in D, the points D and *e* being ſuppoſed to coincide; and let the arches *ce*, *eh* be on different ſides of the baſe, but on the ſame ſide of DE. Then, whether the arches *ce*, *eh* be perpendicular to the baſe and form one continued arch *ch*, or touch the baſe and form a cuſpis of
Fig. 99. the firſt kind at *e*, the point E is a cuſpis of the firſt kind. For,
n. 1. & when PN is above the baſe, VM is to be taken upon VP from
2. V towards P, and when PN is below the baſe, VM is to be taken

taken upon PV produced beyond V, by art. 256. And, conversely, when E is a cuſpis of the firſt kind, if the tangent at E be parallel to the baſe, when the baſe AP increaſes the ordinate from one of the arches CE, HE increaſes and the ordinate from the other decreaſes. The fluxion of one of the ordinates is poſitive and the fluxion of the other is negative; ſo that the right lines which repreſent theſe fluxions muſt be taken on oppoſite ſides of the baſe; and *ce*, *eh* muſt either form a continued arch *ch*, or a cuſpis of the firſt kind at *e*. In both caſes the firſt fluxion of DE vaniſhes; and in the former, the fluxion of D*e*, or the ſecond fluxion of DE, being given, the fluxion of the baſe vaniſhes: In the latter caſe, the ſecond fluxion of DE vaniſhes as well as the firſt fluxion, and, the third fluxion of DE being compared with the fluxion of the baſe, the one vaniſhes when the other is given. In general, when E is a cuſpis of the firſt kind and the tangent at E is parallel to the baſe, the fluxions of DE of any number of ſucceſſive orders may vaniſh with the fluxion of the baſe being given; but the fluxion of the next order to thoſe that vaniſh cannot be to the fluxion of the baſe in any aſſignable ratio, and is ſaid (according to the uſual language on this ſubject) to become infinitely great, in the ſame ſenſe as the ordinate of a curve is commonly ſaid to become infinite when it is ſuppoſed to coincide with an aſymptote.

PROP. XXIII.

270. *The arches ce, eh being on the ſame ſide of* De, Fig. 100. *the point* E *is a cuſpis of the firſt kind when ceh is a continued arch, but of the ſecond kind when e is a cuſpis of the firſt kind and the tangent at e is oblique to* De; *or when e is itſelf a cuſpis of the ſecond kind.*

Caſe 1. When *ceh* is a continued arch and touches the ordinate De, VM is to be taken upon VP from V towards P both when N deſcribes the part *ce* and the part *eh*, by art. 256. The right line Et parallel to DL is the tangent of both the branches CE

CE and EH, by prop. 20. Therefore the point E is a cuſpis. The areas $DecB$, $DefB$, $DebB$ are reſpectively equal to the rectangles contained by the right lines KC, Kf, KH and the given right line DG. Therefore KC is leſs than Kf, and Kf leſs than KH; ſo that the tangent Ef muſt paſs betwixt the arches EC, EH; and, theſe arches being therefore convex towards each other, E is a cuſpis of the firſt kind.

FIG. 101. 271. *Caſe* 2. Let e be a cuſpis of the firſt kind, and if the tangent at e be not parallel to the baſe, the point E ſhall be a cuſpis of the ſecond kind. For let the perpendicular at P meet the arches ce, eb, CE, EH in the points n, N, m and M, the tangent Ef in R, and EK, ef parallel to the baſe in V and S. Then, by art. 256. the right lines VM, Vm are to be taken from V towards P when N deſcribes ce or n deſcribes eb. The rectangles contained by VM, Vm and VR, and by the given right line DG, are reſpectively equal to the areas $DeNP$, $DenP$ and $DeSP$; and the areas $DeNP$, $DenP$ are either both leſs or both greater than $DeSP$. Therefore VM, Vm are both leſs or both greater than VR, the arches EM, Em are on the ſame ſide of the tangent Ef, and, conſequently, E is a cuſpis of the ſecond kind.

272. *Caſe* 3. When e is a cuſpis of the ſecond kind, it appears in the ſame manner that the arches EM, Em are on the ſame ſide of the tangent Ef, and therefore E is a cuſpis of the ſame kind.

273. COR. I. In the firſt caſe, the point E is a cuſpis of the firſt kind, and the right line which meaſures the fluxion of the baſe vaniſhes when that which meaſures the ſecond fluxion of the ordinate is given. The point E is alſo a cuſpis of the firſt kind when the right line which meaſures the ſecond fluxion of the ordinate DE vaniſhes, the fluxion of the baſe being gievn, if the two arches ec, eb are on the ſame ſide of DE and are convex towards each other. But E is not always a cuſpis of this kind when the right line which meaſures the ſecond fluxion of the ordinate vaniſhes, that which meaſures the fluxion of the baſe being given, or when the latter vaniſhes the former being given, though the arches CE, EH be on the ſame ſide of DE. For in this caſe e and E may be each a cuſpis of the ſecond kind.

274. COR.

274. Cor. II. In the ſecond caſe, E is a cuſpis of the ſecond kind, and the right line which meaſures the third fluxion of the ordinate DE vaniſhes when that which meaſures the fluxion of the baſe is given, or the latter vaniſhes the former being given.

275. Cor. III. When the right line which meaſures the ſecond fluxion of the ordinate DE is in an aſſignable ratio to that which meaſures the fluxion of the baſe, and the arches CE, EH are on the ſame ſide of the ordinate, then the point E is always a cuſpis of the ſecond kind, whether the third fluxion of DE be aſſignable or not: for in that caſe the tangent of the arch *ce* at *e* is oblique to D*e*, the point *e* is a cuſpis either of the firſt or ſecond kind; and therefore E is a cuſpis of the ſecond kind, by art. 271. and 272. When the ratio of the right line which meaſures the ſecond fluxion of the ordinate DE to that which meaſures the fluxion of the baſe is not aſſignable, (the arches CE, EH being ſtill on the ſame ſide of DE) if the ratio of the right line which meaſures the third fluxion of DE to that which meaſures the fluxion of the baſe be aſſignable, E is ſtill a cuſpis of the ſecond kind; but when this latter ratio alſo is not aſſignable, E may be a cuſpis of either kind. FIG. 100. & 101.

276. The ordinate DE is a *maximum* or *minimum* of the ſecond kind in all caſes when E is a cuſpis of either kind, that only excepted wherein E is a cuſpis of the firſt kind and the tangent at E is at the ſame time parallel to the baſe. And we may conclude DE to be a *maximum* or *minimum* of this kind, if the curve is not continued on both ſides of the ordinate, not only when the firſt fluxion of the ordinate is to the fluxion of the baſe in an aſſignable ratio, but alſo when the fluxion of the baſe being given the fluxion of the ordinate of any ſubſequent order is to the fluxion of the baſe in an aſſignable ratio. For in all thoſe caſes the point E is a cuſpis of the ſecond kind; or, if it is a cuſpis of the firſt kind, the tangent at E is not parallel to the baſe.

277. What we have ſaid of the greateſt and leaſt ordinates is eaſily applied to the greateſt or leaſt rays that can be drawn from a given point to a curve. Let S be a point that is given in the plane of the curve DPE, SA a right line given in poſition, SP any right line from S that meets the curve in P, PM a perpendicular FIG. 102.

pendicular from P on SA in M. Then, the fluxion of SM being given, the ray SP is a *maximum* or *minimum* when its first fluxion vanishes and its second fluxion does not vanish, or when the fluxions of the subsequent successive orders vanish, if the number of all those fluxions that vanish be an odd number; for if MQ be taken upon MP always equal to SP, the ordinate MQ of the curve IQ shall in those cases be a *maximum* or *minimum* by prop. 21. When the number of the successive fluxions of SP or MQ that vanish is an even number, the point Q is a point of contrary flexure in the curve IQ, and MQ or SP is neither a *maximum* nor *minimum*; but it does not follow, that P is a point of contrary flexure in the curve DPE.

278. To illustrate by an obvious example the necessity of having regard to those limitations in resolving by the common rules the problems that relate to the *maxima* and *minima*, let DPE be a common parabola, DK its axis, S a point given within the curve, SA be perpendicular to the axis DK in K, and PN perpendicular to it in N. Then, SM or PM being supposed to flow uniformly, let the fluxions of SP be computed. The first fluxion of SP vanishes when SP becomes perpendicular to the curve, and by the common rule SP ought in that case to be a *maximum* or *minimum*. But it is certain, that if DK be equal to the sum of 3DN and one half of the parameter of the axis, and the point S be not upon the axis, the right line SP, though perpendicular to the curve, is neither a *maximum* nor *minimum*; for a circle described from the center S through P falls without the arch PD and within the arch PE, as shall be demonstrated afterwards. In this case it will be found, that the second fluxion of SP vanishes as well as its first fluxion, but that its third fluxion does not vanish; so that, according to prop. 21. SP is not a *maximum* or *minimum* in this case, though its first fluxion vanish. If the third fluxion of SP also vanish, then SP is a *maximum* or *minimum* by the same proposition. And this is the case when the point S is upon the axis at *f* and *f*D is equal to one half of the parameter; for, in this case, when P comes to D, the first, second and third fluxions of *f*P vanish, but its fourth fluxion does not vanish, as will appear by the computation; and it is known that *f*D is the least

leaſt right line that can be drawn from the point f to the parabola. Nor is there any curve but the circle alone that does not afford examples of this kind. It may be of uſe for abridging computations of this ſort to obſerve, that when the value of any quantity z is expreſſed by a term that is formed from any power of another quantity u and invariable quantities, if while u is finite the firſt fluxion of u and its fluxions of any ſubſequent ſucceſſive orders vaniſh, the fluxions of z of the ſame orders vaniſh at the ſame time. If z be expreſſed by a fraction whoſe numerator N and denominator D are both finite when the fluxion of N is to the fluxion of D as N is to D, then the fluxion of z vaniſhes; and if the fluxions of N and D of any ſubſequent ſucceſſive orders be to each other reſpectively in the ſame ratio of N to D, then the fluxions of z of the ſame orders vaniſh.

279. Let us now reſume the conſtruction of the 211th and 212th articles, where P the interſection of LP the tangent of any curve LB and of SP the perpendicular from the given point S is ſuppoſed to be always found in the curve DPE; and where it is ſhewn, that if the angle SPT be made equal to SLP, PT ſhall be the tangent of the curve DPE. Let ST be always perpendicular from S upon this tangent PT, and T be always found in the curve GTH; and, if AS be any right line given in poſition that produced beyond S meets PT produced beyond T in Q, the angles ASL, ASP, AST ſhall be in arithmetical progreſſion, and the angle AST ſhall be equal to 2ASP—ASL. The angle ASP being ſuppoſed to increaſe, the angle SQT increaſes or decreaſes according as the arch of the curve deſcribed by P is concave or convex towards S; and, becauſe AST (or 2ASP—ASL) is equal to SQT added to the right angle STQ, it follows, that, the fluxion of the angle ASP being ſuppoſed poſitive, the arch deſcribed by P is concave or convex towards S according as the fluxion of 2ASP—ASL is poſitive or negative, that is, according as the fluxion of 2ASP is more or leſs than the fluxion of ASL. When the arch deſcribed by L is convex towards S, ASL decreaſes while ASP increaſes, the fluxion of ASL is negative, the angle AST increaſes, and the arch deſcribed by P is always concave towards S. Theſe things were mentioned above, but without a proof. Fig. 103.

FIG. 104. 280. When the point P in the curve DE is a point of contrary flexure, the point T in the curve GTt is a cuſpis, the angle AST (or 2ASP—ASL) is a *maximum* or *minimum*; and the right line ST is alſo a *maximum* or *minimum*, unleſs when the angle SPT or SLP is a right one, or when S and P coincide. For let the angle STK be made equal to SPT ſo that TK may ly the ſame way from ST as PT from SP, and TK ſhall be the tangent of the curve GTt at T, by art. 211. & 212. Let Pρ be the arch terminated at P that is concave towards S, and Pp the arch that is convex towards it, and while P deſcribes the arches Pρ, Pp, let T deſcribe the arches Tτ, Tt. It is manifeſt, that theſe arches Tτ, Tt are on the ſame ſide of the right line PT; and, ſince they have the ſame tangent TK, the point T is a cuſpis. The angle SQT is a *maximum* or *minimum* when P is a point of contrary flexure, and therefore AST (or 2ASP—ASL) is a *maximum* or *minimum*. When the angle STK or SPT is not a right one, the right line ST either exceeds the right lines that can be drawn from S to the adjoining parts of either arch Tτ, Tt, or is leſs than them, and therefore ST is a *maximum* or *minimum*.

281. It follows, converſely, that the curve pp being continued on both ſides of the right line PL, and the tangent at P being oblique to SP, if ST be a *maximum* or *minimum*, the point P in the curve DPE is a point of contrary flexure. Therefore, ƒP an arch of a circle deſcribed from the center S through P being ſuppoſed to flow uniformly, if the fluxion of ST vaniſh, the point P is a point of contrary flexure, the ſame limitations being underſtood as were deſcribed in prop. 21. Let the invariable fluxion of the arch ƒP be repreſented by a given line PI taken upon PL, and let IH perpendicular to PI meet TP produced in H; then ſhall PH and IH repreſent the fluxions of the curve DP and ray SP by prop. 16. & 17. The rectangle contained by SP and PI is equal to the rectangle contained by ST and PH; and therefore, when the fluxion of ST vaniſhes, the rectangle contained by IH (which meaſures the fluxion of SP) and PI is equal to the rectangle contained by ST and the right line which meaſures the fluxion of PH, and the fluxion of PH is to the fluxion of SP as PI is to ST or as PH is to SP.

But, becauſe PI is ſuppoſed to be invariable, it follows from prop. 15. that the fluxion of IH is to the fluxion of PH as PH is to IH. Therefore the fluxion of IH is to the fluxion of SP (which is expreſſed by IH) as the ſquare of PH is to the rectangle contained by SP and IH; and, conſequently, the right line which meaſures the ſecond fluxion of SP, PH which meaſures the firſt fluxion of the curve, and the ray SP are in continued proportion; which coincides with one part of the rule that is uſually given for finding the points of contrary flexure in curves, that are conſidered as ſpirals, and are defined by an equation that expreſſes the relation of the fluxions of fP, SP or DP to each other. But this rule is not to be admitted without the limitations that follow from prop. 21. though the firſt fluxion of ST vaniſh. If the curve deſcribed by P be not continued on both ſides of the right line PL, or if the fluxions of ST of the ſubſequent ſucceſſive orders vaniſh, and the number of all its fluxions that vaniſh be an even number, we cannot conclude that P is a point of contrary flexure. When the firſt fluxion of ST vaniſhes, its ſecond fluxion vaniſhes alſo (the fluxion of the arch fP being invariable) when the ratio of the ſecond fluxion of PH to the ſecond fluxion of SP is the ſame as that of PH to SP; and when the fluxions of PH of any ſucceſſive orders from the firſt, are to the fluxions of SP of the ſame orders, reſpectively, in the ſame ratio of PH to SP, then the fluxions of ST of theſe orders alſo vaniſh.

282. It follows alſo from the 280th article, that, the curve pp being continued on both ſides of the right line PL and the angle SPT being acute, P is a point of contrary flexure, when the fluxion of the angle AST vaniſhes, or when the fluxion of the angle ASP becomes equal to one half of the fluxion of ASL, (thoſe caſes however being excepted in which AST is not a *maximum* or *minimum* according to prop. 21.) that is, when the angular velocity of SP about S is one half of the angular velocity of SL about S, and is in the ſame direction; for if the motion of SL and SP be in different directions, the angular motion of ST does not vaniſh when the angular motion of SP is equal to one half of the angular motion of SL, but on the contrary is equal to the ſum of the motion of SL added to twice-

twice the motion of SP. The point T in the curve GTH is a point of contrary flexure, for the ſame reaſon, when the fluxion of AST becomes equal to one half of the fluxion of ASP, or when the fluxion of ASP becomes equal to two thirds of the fluxion of ASL, becauſe ASL, ASP, AST are in arithmetical progreſſion. In general, the ſeries of curves BL, DP, GT, *&c.* being continued, each of which is ſuppoſed to paſs through the interſections of the tangents of the preceeding curve with the perpendiculars from S on theſe tangents, that point in the laſt curve of the ſeries which correſponds to L in the firſt curve is a point of contrary flexure when the fluxion of the angle ASL is to the fluxion of ASP as the number of curves in the ſeries is to the ſame number diminiſhed by unit; and that point is a cuſpis when the fluxion of ASL is to the fluxion of ASP as the number of curves is to the ſame number diminiſhed by two.

283. In this laſt manner the points of contrary flexure and reflexion are ſometimes eaſily diſcovered. For example, when ALB is a circle, C the center, S any point within the circle, AB the diameter that paſſes through S, the point P in the curve DPE is a point of contrary flexure when the ſquare of LS is one third part of the rectangle ASB, or (LS being produced till it meet the circle again in Z) when LS is one fourth part of LZ. For, let CV be perpendicular on LZ in V; and, SLR being made equal to CLQ if LR meet the diameter in R, the fluxion of ASL ſhall be to the fluxion of ACL (or ASP) as CR is to SR (by prop. 18.) or as LV is to LS; and therefore, when LS is equal to one half of LV, or one fourth part of LZ, the fluxion of ASP is one half of the fluxion of ASL, and P is a point of contrary flexure; that caſe being excepted wherein CS is equal to SB, and SB is one fourth part of AB, in which the point L coincides with B when the fluxion of ASP becomes equal to one half of the fluxion of ASL, and the curve deſcribed by P is not continued on both ſides of the tangent at B. When S is nearer to the center of the circle than to the circumference, or when S is without the circle, or upon the circumference, the fluxion of ASP never becomes equal to one half of the fluxion of ASL. When S is without the circle, the angular velocity of SP may become equal to one half of the angular velocity

FIG. 105.

of

of SL; but the motions of SP and SL are in oppoſite directions when this happens, the fluxion of ASP being ſuppoſed poſitive the fluxion of ASL is negative, and the fluxion of AST does not vaniſh. The curve ALB being ſtill a circle, and the ſeries of curves ALB, DPE, GTH, &c. being continued, the point in the laſt curve of the ſeries that correſponds to L in the circle is a point of contrary flexure when SL is to SZ as the number of curves diminiſhed by unit is to the ſame number increaſed by unit, or when the ſquare of SL is to the given rectangle ASB in that ratio; and that point is a cuſpis when SZ is to SL as the number of curves is to this number diminiſhed by two; the caſe being always excepted when S is ſo ſituated that the point L coincides with B when this happens.

284. In like manner theſe points are readily determined in many other curves, eſpecially when the curve BL is ſuch that a point can be aſſigned, from which rays being drawn to the curve and perpendiculars to the tangents of the curve, the angular velocity of the ray about that point is to the angular velocity of the perpendicular in any invariable ratio: When BL is any curve of this kind, it has no point of contrary flexure; and if S coincide with that given point, the curves DPE, GTH being of the ſame kind *, have alſo no point of contrary flexure; but if S be any other point in the plane of the curve BL, the curves DPE, GTH, &c. may have points of contrary flexure and reflexion that are often eaſily determined by art. 282. & 210. When BL is a parabola and S is within the parabola upon the axis, P is a point of contrary flexure when LQ is equal to ES, or when BM (LM being perpendicular to the axis in M) is one third part of BS. The curve deſcribed by T in this caſe has two points of reflexion correſponding to the two points of contrary flexure in the curve DP, and a third point of reflexion upon the axis of the parabola at a diſtance from S towards A equal to one fourth part of the parameter. FIG. 106.

285. The right line ST is alſo a *maximum* or *minimum*, if the fluxion of fP vaniſh when the fluxion of ST is ſuppoſed to be aſſignable, by art. 245. and in this caſe the ratio of the difference betwixt the rectangle PHK and the rectangle contained by FIG. 104.

* Deſcript. curvarum, part. 2. prop. 14, 16, &c.

SP

SP and the right line which meaſures the fluxion of PH to the ſquare of PH by increaſing becomes unaſſignable; and this coincides with the ſecond part of the rule that is commonly given for finding the points of contrary flexure in curves that are conſidered as ſpirals. It is however to be allowed with limitations analogous to thoſe above mentioned in the 245th, 262d and 267th articles. But having inſiſted on this ſubject at a ſufficient length, we proceed to conſider ſome other affections of curve lines.

CHAP. X.

Of the Aſymptotes of curve Lines, the Areas bounded by them and the Curves, the Solids generated by thoſe Areas, of ſpiral Lines, and of the Limits of the Sums of Progreſſions.

286. A Right line given in poſition is an *Aſymptote* of the branch of a curve when they never meet, but approach to each other continually, ſo that by producing them their diſtance from each other becomes equal to any line how ſmall ſoever that may be given; and the branch of the curve that approaches thus to the aſymptote is ſaid to be of the *hyperbolic* kind. A branch of a curve that approaches in the ſame manner to a parabola is ſaid to be of the *parabolic* kind and to have a parabolic aſymptote, of which there may be as many different kinds as there are parabolas of different orders. In general, any curve lines may be ſaid to be aſymptotes of each other mutually when they approach to each other in this manner.

287. In the common doctrine of the hyperbola it is ſhewn, that a curve and a right line may continually approach to each other in this manner and never meet. Let CO and CV be the right lines that are called the aſymptotes of the hyperbola *a*ME, *a*K and IM two right lines parallel to CV bounded by the aſymptote CO and the curve in *a* and K, I and M; and IM

Fig. 107.

ſhall

Pl. XI. Pa. 240.

Fig. 93.

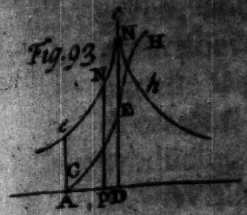


Fig. 94.

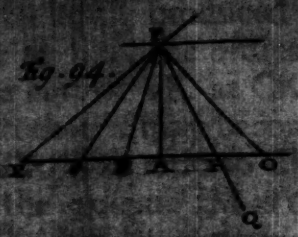


Fig. 95.

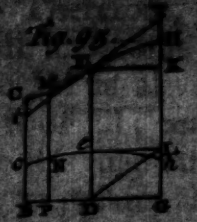


Fig. 96.

Fig. 97.

Fig. 99. n. 2.

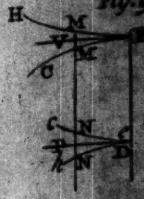


Fig. 99. n. 1.

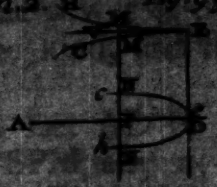


Fig. 98.

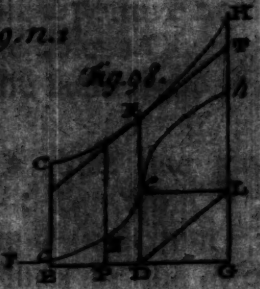


Fig. 100.

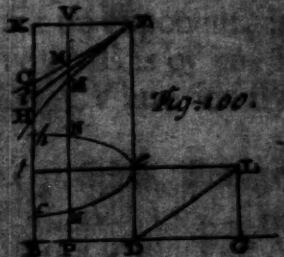


Fig. 101.

Fig. 92.

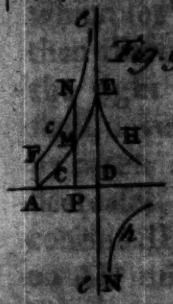


Fig. 103.

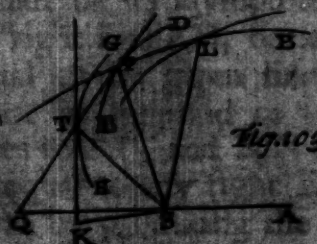


Fig. 104.

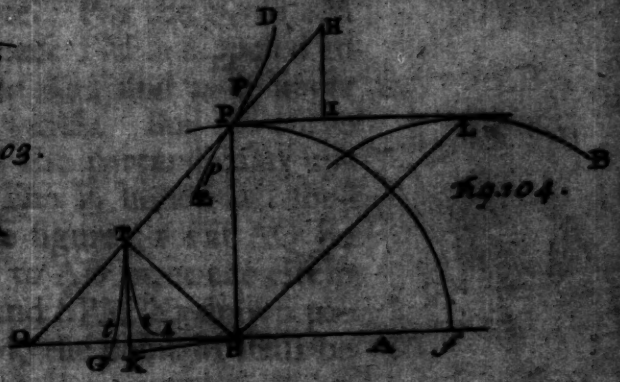


Fig. 105.

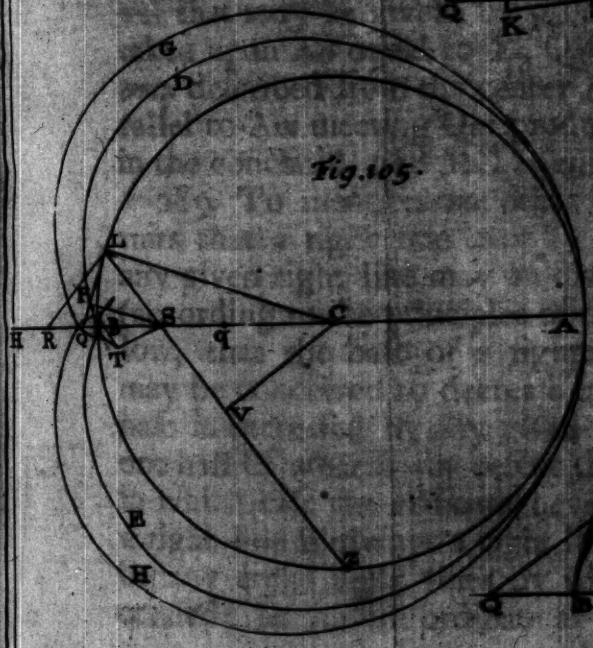


Fig. 106.

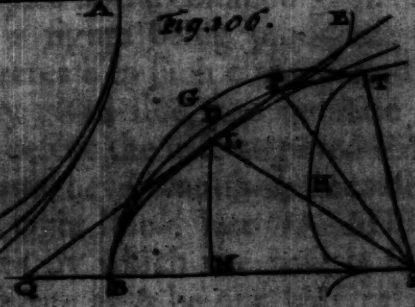
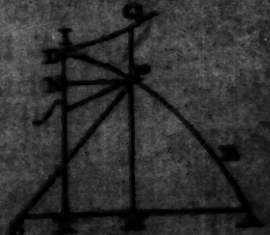
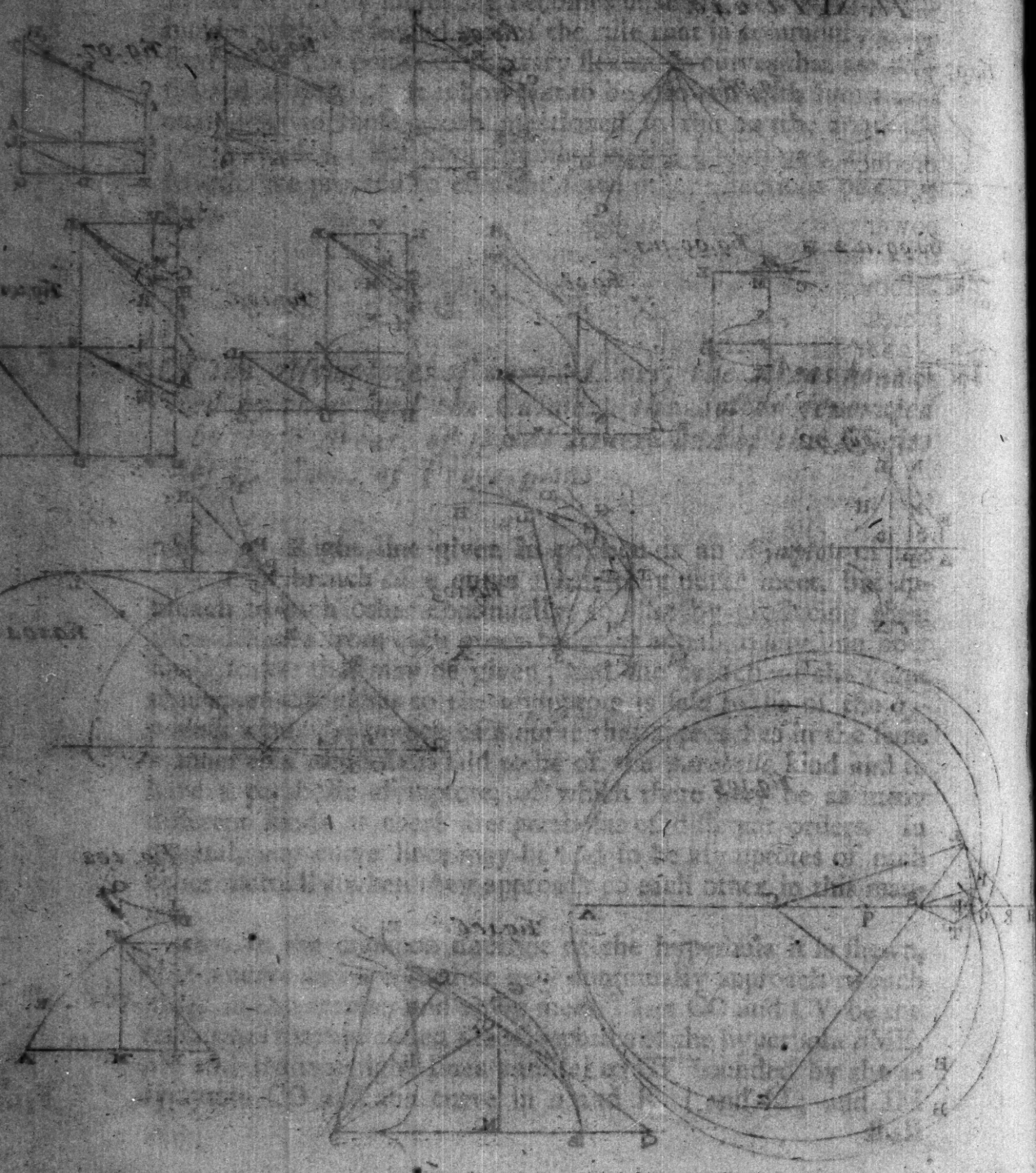


Fig. 102.





ſhall be to σK as CK is to CI. Therefore, CK and σK being given, and CI being produced till it become any multiple how great ſoever of CK, the ordinate IM is always aſſignable, being the ſame part of σK as CK is of CI; and IM continually decreaſes, ſo that it may become leſs than any given line Z by producing CI till its ratio to CK be greater than that of σK to Z. When the ordinate IM is to σK as the ſquare, cube, or any power of CK that has a poſitive number for its exponent, is to the ſame power of the abſciſs CI, the curve σM is of the hyperbolic kind, and the ſame right lines CO, CV are its aſymptotes.

288. Let AD be a right line given in poſition, S a given point, SPM any right line from S meeting AD in P; upon which let PM be taken always from P equal to a given line $A\alpha$; then the curve σME ſhall be the *conchoid* of the ancients, and the right line AD ſhall be its aſymptote. For the curve σM ſhall never meet this right line AD, becauſe PM is ſuppoſed to be taken always equal to the given line $A\alpha$; and as it never decreaſes, ſo it cannot be ſuppoſed to vaniſh. But the curve continually approaches to AD, and MN the perpendicular upon AD may become equal to any right line Z, how ſmall ſoever that may be given, by producing the figure; for if AQ be taken upon $A\alpha$ equal to Z, Qm parallel to AD meet the circle $\gamma m\varkappa$ deſcribed from the center A in m, and SPM be drawn parallel to Am meeting Qm produced in M, the point M ſhall be in the conchoid, and MN equal to AQ ſhall be equal to Z. Fig. 108.

289. To mention one other ſimple inſtance: Whoever admits that a right line may be continued at pleaſure, and that any given right line may be divided into two equal right lines, (according to the principles of the common geometry,) will allow, that the baſe of a figure being produced, the ordinate may be conceived to decreaſe in ſuch a manner, that when the baſe is increaſed by any given line, the ordinate may become one half of what it was before the baſe acquired that increment; in which caſe the ordinate never vaniſhes, becauſe the half of a right line is always a right line. Let OA, AD, DE, EF, &c. be any equal right lines by which the baſe is ſucceſſively increaſed; let $A\alpha$ the ordinate at A be one half of $O\omega$ the ordi- Fig. 44.

nate at O, Dd one half of Aa, Ee one half of Dd, and ſo on. And it is manifeſt, that the logarithmic curve (deſcribed in art. 176.) which paſſes through the points o, a, d, e, f, and the extremities of all the ordinates in the ſame ſeries, which may be continued at pleaſure, can never meet the baſe. In general, any quantity may be conceived to decreaſe continually and yet never be quite exhauſted; as, when the right line AP that touches the circle Agr in A is produced and KP is always joined, the arch rg, the angle rKg, and the perpendicular gf decreaſe continually, ſo that they may become leſs than any given quantity of the ſame kind by producing AP to an aſſignable diſtance, but never vaniſh.

FIG. 107.

290. We have mentioned thoſe ſimple inſtances, to ſhew that there is nothing ſo abſtruſe or inconceivable in what Geometricians demonſtrate concerning the aſymptotes of curve lines as is ſometimes repreſented. They are under no neceſſity of ſuppoſing, that * *a finite quantity or extenſion conſiſts of parts infinite in number*, or that there are any more parts in a given magnitude than they can conceive and expreſs: It is ſufficient that it may be conceived to be divided into a number of parts equal to any given or propoſed number, and this is all that is ſuppoſed in ſtrict geometry concerning the diviſibility of magnitude. It is true, that the number of parts into which a given magnitude may be conceived to be divided is not to be fixed or limited, becauſe no given number is ſo great but a greater than it may be conceived and aſſigned; but there is not therefore any neceſſity for ſuppoſing that number infinite; and if ſome may have drawn very abſtruſe conſequences from ſuch ſuppoſitions, geometry is not to be loaded with them.

291. Though Geometricians are under no neceſſity of ſuppoſing a given magnitude to be divided into an infinite number of parts, or to be made up of infinitеſimals, they cannot ſo well avoid the ſuppoſing it to be divided into a greater number of parts than may be diſtinguiſhed in it by ſenſe in any particular determinate circumſtances. But they find no difficulty in conceiving this; and ſuch a ſuppoſition does not appear to be repugnant to

* Princip. of human knowledge, § 124.

the

the common ſenſe of mankind, but on the contrary to be moſt agreeable to it, and to be illuſtrated by common obſervation. It would ſeem very unaccountable, not to allow them to conceive a given line, of an inch in length for example, viewed at the diſtance of ten feet, to be divided into more parts than are diſcerned in it at that diſtance; ſince by bringing it nearer a greater number of parts is actually perceived in it. Nor is it eaſy to limit the number of parts that may be perceived in it when it is brought near to the eye, and is ſeen through a little hole in a thin plate, or when by any other contrivance it is rendred diſtinct at ſmall diſtances from the eye. If we conceive a given line that is the object of ſight to be divided into more parts than we perceive in it, it would ſeem that no good reaſon can be aſſigned why we may not conceive tangible magnitude to be divided into more parts than are perceived in it by the touch, or a line of any kind to be divided into any given number of parts, whether ſo many parts be actually diſtinguiſhed by ſenſe, or not. If the hyperbola and its aſymptote were accurately deſcribed, they would ſeem to ſenſe to join each other, at various diſtances from the center according to the different circumſtances in which they might be perceived; but we may conceive the ordinate at the point where they ſeem to join to have a real magnitude, in the ſame manner as we conceive a given line to ſubſiſt when it is carried to ſo great a diſtance that it vaniſhes to ſight, or any ſmall particle (as an atom in the ſun-beams) to exiſt, though it eſcape the touch, or have no tangible magnitude. It may perhaps illuſtrate this, if it be conſidered, that the curve cannot be ſaid to meet its aſymptote in this caſe, in the ſame ſenſe that a circle is ſaid to meet its diameter, which it appears to interſect in all caſes, whatever the diſtance or poſition of the figure, or the acuteneſs of the ſenſe may be; whereas the ordinate of the hyperbola that vaniſhes to ſight at a greater diſtance, becomes viſible at a leſs diſtance, and may be diſtinguiſhed into more and more viſible parts, in proportion as it approaches to the eye, or the ſenſe is more acute. And ſurely it muſt be allowed that there is ground for a difference between a line that eſcapes the ſight and vaniſhes, becauſe of its diſtance from the eye, and a

line that in no caſe can ever be perceived, or can be ſuppoſed to have any exiſtence. Perhaps it will be ſaid by ſome, that, ſtrictly ſpeaking, it is not the ſame line that in thoſe different circumſtances has a greater and leſs number of viſible parts. In anſwer to this, it is ſufficient for our purpoſe to obſerve, that as there can hardly be any Philoſopher but will allow that there is ſome ſenſe in which it is the ſame inch-line that has more viſible parts at eight inches diſtance from the eye than when it is held at the length of the arm; ſo it is not incumbent on us to explain in what ſenſe this is to be underſtood according to every ſcheme: It is enough that this ſenſe muſt be ſuppoſed to be plain and obvious, as it is univerſal, and that Geometricians ought to be allowed to conſider lines and figures in this ſenſe as well as every body elſe. Philoſophers and the vulgar equally conceive the ſun and planets, and the other objects of their obſervation and enquiries, to be the ſame bodies, when ſeen at different diſtances or different times: And if they were not allowed to conſider thoſe bodies as made up of more parts than are perceived by ſenſe, and Geometricians were under the ſame limitations as to magnitude in general, they would not be a little perplexed; nor is it the more intricate and ſubtile part of thoſe ſciences only that would be thus pared off. The learned author abovementioned tells us, "That the magnitude of the object which "exiſts without the mind, and is at a diſtance, continues al-"ways invariably the ſame *." He ſeems to ſpeak of tangible magnitude. It is not our buſineſs here to enquire how, according to his doctrine, tangible magnitude can be conceived to exiſt without the mind any more than viſible magnitude. This conceſſion perhaps is made only for the ſake of his Argument in this place; but the evidence for the exiſtence of ſuch an object may very well be ſuppoſed to approach to that which we have for the exiſtence of any other objects that are not immediately perceived by us. And ſince he admits it, and argues from it, in this treatiſe, it would ſeem that ſome invariable magnitude is to be allowed, which † we apprehend by

* New Theory of Viſion, § 55.

† Ibid. § 54.

the

the ſight, though not immediately; and that this magnitude may be conceived to be divided into any given number of parts, from the demonſtrations propoſed by Geometricians on this ſubject. In applying which, it ought to be remembred, that a ſurface is not conſidered by them as a body of the leaſt ſenſible magnitude, but as the termination or boundary of a body; a line is not conſidered as a ſurface of the leaſt ſenſible breadth, but as the termination or limit of a ſurface: nor is a point conſidered as the leaſt ſenſible line, or a moment as the leaſt perceptible time; but a point as a termination of a line, and a moment as a termination or limit of time. In this ſenſe they conceive clearly what a ſurface, line, point *, and a moment of time is; and the *poſtulata* of EUCLID being allowed and applied in this ſenſe, the proofs by which it is ſhewn that a given magnitude may be conceived to be divided into any given number of parts, appear ſatisfactory: And if we avoid the ſuppoſing the parts of a given magnitude to be infinitely ſmall, or to be infinite in number, this ſeems to be all that the moſt ſcrupulous can require.

292. But to proceed: The arch Ag, the ſector AKg, and the ordinate PM increaſe continually while AP is produced; but the arch Ag never amounts to Ax, the ſector AKg to the ſector AKx, or the ordinate PM to the given right line AK the diſtance betwixt the baſe AP and aſymptote KI. An area APNF may increaſe at the ſame rate as the arch Ag, the ſector AKg, or the ordinate PM increaſes; and, by flowing in the ſame manner, it may approach in magnitude to a given ſpace continually while the figure is produced, and never amount to it. Let us reſume, for an example, the conſtruction of art. 289. and, *Oo* being biſected in M, OM in N, ON in R, and ſo on, let the rectangles O*o*BA, A*a*HD, D*d*IE, E*e*KF, &c. be completed, and let N*d*, R*e* meet A*a* in L and V. The rectangles AH, DI, EK, &c. are reſpectively equal to the rectangles *oa*, ML, NV, &c. and the ſum of thoſe rectangles being ſucceſſively equal to *oa*, *o*L, *o*V, &c. it is therefore always leſs than the rectangle *o*A, though the ſeries of thoſe rectangles be conti-

FIG. 107.

FIG. 44.

* The Analyſt, § 35.

nued

nued till their number become equal to any given number how great ſoever it may be. The differences betwixt their ſucceſſive ſums and the rectangle *o*A are ſucceſſively equal to the rectangles AM, AN, AR, &c. which continually decreaſe, and by continuing the ſeries may become leſs than any given ſpace. Therefore the rectangle *o*A is the limit to which the ſum of the rectangles AH, DI, EK, &c. continually approaches while the figure is produced, and to which it never amounts unleſs the figure be ſuppoſed to be infinitely produced. As for the area bounded by the curve *adef* and the baſe AF, it is always leſs than the ſum of the rectangles AH, DI, EK, &c. and therefore is always leſs than the rectangle *o*A. It approaches to a leſſer limit, *viz.* the rectangle contained by A*a* and O*o*, if we ſuppoſe *o* to be the point where the tangent makes an angle with the ordinate that is half a right one; and the limit of the area bounded by the curve *oadef*, ordinate O*o* and baſe OF is the ſquare of O*o*. For let any ordinate P*p* meet *o*B in Z, and the fluxion of P*p* (or of *p*Z) ſhall be to the fluxion of the baſe OP as P*p* is to O*o*, by art. 176. Therefore the fluxion of the area OP*po* is equal to the fluxion of the rectangle contained by O*o* and *p*Z, by prop. 4. and the area OP*po* is equal to this rectangle by art. 24. And as *p*Z is always leſs than O*o*, but approaches to it ſo that the difference P*p* may become leſs than any given line by producing the figure, ſo the area OP*po* is always leſs than the ſquare of O*o*, but approaches to it continually, ſo that their difference (the rectangle contained by P*p* and O*o*) may become leſs than any given ſpace.

FIG. 107. 293. The right line KI parallel to the baſe AP being the aſymptote of any curve *a*ME, let PN the ordinate from the curve FN*e* be always to the given right line DG as the fluxion of the ordinate PM is to the fluxion of the baſe, and let PM meet *ap* parallel to the baſe in *p*. Then the baſe AP ſhall be the aſymptote of the curve FN*e*, and the area APNF ſhall be always leſs than the rectangle contained by K*a* and DG, though the baſe AP and curve FN be produced to any diſtance how great ſoever, but ſhall continually approach to that rectangle, ſo that their difference may become leſs than any given ſpace by producing the figure. For it appears, as in art. 248. that the area APNF

APNF is always equal to the rectangle contained by *p*M and the given right line DG, and therefore is always leſs than the rectangle contained by K*a* and DG by a ſpace equal to the rectangle contained by IM and DG, which may become leſs than any given ſpace, but never vaniſhes; becauſe IM may become leſs than any given right line, but never vaniſhes, ſince KI is ſuppoſed to be the aſymptote of the curve *a*ME.

294. When CV perpendicular to CO is alſo an aſymptote of the curve EM*a* produced beyond *a*, the right line C*v* (which is the continuation of CV) is an aſymptote of the curve deſcribed by N continued on the other ſide of AF; and, if the ordinates PM, PN meet the baſe on the other ſide of A, the area APNF ſhall be ſtill equal to the rectangle contained by *p*M and DG, and will in this caſe exceed any given ſpace by continuing the curve FN alongſt the aſymptote C*v*, becauſe *p*M will exceed any given right line by continuing the curve *a*M alongſt the aſymptote CV. In the former caſe, when the curve FN was produced alongſt the baſe AD, which is one of its aſymptotes, the area APNF was always leſs than a certain ſpace, (the rectangle contained by K*a* and DG,) which we therefore call its limit. In the latter caſe, when the curve FN is produced alongſt its other aſymptote C*v*, the area APNF may exceed any given ſpace, and has no aſſignable limit. They who ſcruple not to ſuppoſe the curve and aſymptote to be infinitely produced, ſay, that the area APNF then becomes equal to its limit in the former caſe, and that it becomes infinitely great in the latter caſe. And this area has been ſaid in certain caſes to be more than infinite by ſome authors, from an analogy they imagined to be betwixt what is negative, nothing, and finite, and what is finite, infinite, and more than infinite.

295. But when the curve EM*a* continued on the other ſide of A touches the right line CV in L, the right line C*v* is ſtill an aſymptote of the curve deſcribed by N continued on the other ſide of F, becauſe the ratio of the fluxion of PM to the fluxion of the baſe, or of PN to DG, may exceed any given ratio while P deſcribes AB; but in this caſe the area APNF (which is equal to the rectangle contained by *p*M and DG) is always leſs than a certain aſſignable ſpace, *viz.* the rectangle con- FIG. 109.

contained by *b*L and DG, (*pa* being ſuppoſed to meet CV in *b*,) becauſe *p*M is always leſs than *b*L. In this caſe, the curve FN being continued alongſt both the aſymptotes BD, B*v* at pleaſure, the area bounded by the two ordinates PN, PN that are on different ſides of AF, is always leſs than the rectangle contained by CL and DG, to which it approaches however as its limit when the figure is produced continually both ways. And when the curve is ſuppoſed to be infinitely produced both ways, and the aſymptotes BD, B*v* are alſo ſuppoſed to be both infinitely produced, the whole area included betwixt the curve and its two aſymptotes is ſaid to become equal to this rectangle contained by CL and DG.

FIG. 107. & 109. 296. While the point N deſcribes the branch of the curve FN that proceeds alongſt the baſe AP, which is one of its aſymptotes, if P*n* be to DG as the fluxion of PN is to the fluxion of the baſe AP, the area AP*nf* ſhall be always equal to the rectangle contained by DG and the exceſs of AF above PN; and the rectangle contained by AF and DG is the limit to which the area AP*nf* continually approaches by producing the curve *fn* and baſe AP. And if a ſeries of curves be deduced in this manner, (the ordinate of any curve being to the given right line DG as the fluxion of the ordinate of the preceeding curve in the ſeries is to the fluxion of the baſe,) the rectangle contained by the ordinate at A of any curve of the ſeries and by the right line DG ſhall be the limit to which the area of the ſubſequent curve continually approaches while it is produced alongſt the baſe. While the point N deſcribes the branch that proceeds alongſt the aſymptote B*v*, the point *n* deſcribes a curve that has the ſame aſymptote; but the area AP*nf* in this caſe may exceed any given ſpace by producing the curve *fn*, and has no aſſignable limit: and the ſame is to be ſaid of the area of any ſubſequent curve in the ſeries.

FIG. 110. 297. When the curve *a*ME has not an aſymptote parallel to the baſe AP, but the angle *k*MT formed by MT the tangent at M and M*k* parallel to the baſe decreaſes ſo that by producing the curve it may become leſs than any given rectilineal angle, but never vaniſhes, the baſe AP is ſtill an aſymptote of the curve deſcribed by N, becauſe the ratio of PN to DG (which is the

ſame

ſame as that of *k*T to M*k* by prop. 14.) may become leſs than any given ratio, and yet never vaniſhes. In this caſe the area APNF may exceed any given ſpace by producing the curve and baſe, becauſe *p*M may exceed any given right line. When CV is an aſymptote of the curve EM*a* produced on the other ſide of *a*, C*v* is an aſymptote of the curve deſcribed by N produced on the other ſide of F; and, PN being on the other ſide of AF, the area APNF may exceed in this caſe alſo any given ſpace by producing the curve along the aſymptote C*v*. Thus, when *a*ME is the logarithmic curve deſcribed in art. 176. *b*V its aſymptote, *ba* the ordinate whoſe logarithm is nothing, the curve FN is the common hyperbola, becauſe the fluxion of PM is to the fluxion of BP (or *bp*) as *ba* is to BP, (by art. 176.) and, PN being to DG in the ſame ratio, the rectangle BPN is equal to the invariable rectangle contained by *ba* and DG. It follows from the geneſis of the logarithmic, that *p*M may exceed any given right line by producing the curve on either ſide of A*a*, and therefore the area APNF may exceed any given ſpace by producing the hyperbola on either ſide of AF: The ſame appears from the common doctrine of the hyperbola. The curve *fn* (art. 293.) in this caſe, and all the ſubſequent curves of the ſeries, are hyperbolas of an higher order; and when P is upon the ſame ſide of A with D, the area terminated by the curve and baſe and the ordinates at A and P has an aſſignable limit which it can never exceed, the ſame that was defined in the laſt article; but when P is on the other ſide of A, betwixt A and B, that area may exceed any given ſpace by producing the curve.

298. The converſe of the 293d article eaſily appears. Let the baſe AD be an aſymptote of the curve FN, and *ap* parallel to the baſe through any given point in the perpendicular at A meet PN in *p*; let the rectangle contained by *p*M and the given right line DG be always equal to the area APNF; let the rectangle *ad* (K*d* being equal to DG) be the limit to which the area APNF continually approaches ſo that their difference becomes leſs than any given ſpace by producing the figure, if any ſuch limit is aſſignable; and let *a*K be taken on the ſame ſide of *ap* with *p*M: then a right line through K parallel to the FIG. 107.

baſe ſhall be the aſymptote of the curve αME. For the rectangle contained by pM and DG ſhall increaſe in the ſame manner as the area APNF (to which it is always equal) increaſes: and as this area approaches continually to the rectangle contained by Kα and DG in ſuch a manner that their difference becomes leſs than any given ſpace by producing the figure APNF; ſo the ordinate pM approaches continually to αK or pI, and their difference IM becomes leſs than any given right line by producing the figure αMIK. Therefore KI is the aſymptote of the curve αME. If no ſuch limit of the area APNF can be aſſigned, but this area may exceed any given ſpace, then the branch αME is not of the hyperbolic kind, and has not a rectilineal aſymptote. For in this caſe pM may exceed any given right line by producing the figure αMp; and, (kt parallel to pM being ſuppoſed to meet the tangent Mt in t and Mk equal and parallel to DG in k,) ſince PN, or kt, may become leſs than any given right line, the angle kMt may become leſs than any given rectilineal angle by producing the curve.

299. When the ordinate PN is reciprocally as any power of BP whoſe index is greater than unit, a limit of the area APNF can be aſſigned, and the curve αME has an aſymptote parallel to the baſe. If the ordinate PN be reciprocally as the ſquare of BP, and the curve FN with the baſe AP be ſuppoſed to be infinitely produced, the ordinate PN at an infinite diſtance is ſaid to be an infiniteſimal of the ſecond order, becauſe PN is to AF as the ſquare of BA is to the ſquare of BP: And the element of the baſe AP being ſuppoſed an infiniteſimal of the firſt order, then ſince the element of PM is to the element of AP as PN is to DG, it follows (according to the doctrine of infiniteſimals) that the element of PM in this caſe becomes an infiniteſimal of the third order. If PN be reciprocally as any higher power of BP, the element of PM becomes an infiniteſimal of an order ſtill lower, AP being ſtill ſuppoſed infinite: And it is propoſed as a rule *, That, the baſe being infinitely produced, if the element of the ordinate becomes an infiniteſimal two or more degrees beneath the element of the baſe, then we may conclude

* Geometr. de l'infini, § 950. &c.

that

that the curve has an aſymptote parallel to the baſe. But, ſince it is acknowledged that we may be led into miſtakes by this rule, unleſs regard be had to the ratio of PM to AP when AP is ſuppoſed infinite, (as when *a*ME is a cubic parabola perpendicular to *ap* in *a*, and the ordinate is ſuppoſed infinite, its element is an infiniteſimal two degrees beneath that of the baſe, becauſe in this curve the element of *p*M is to the element of *ap* as a given ſquare is to the triple ſquare of *p*M, and yet this curve has no aſymptote;) and when this ratio is known, it may be diſcovered eaſily from thence if the curve has an aſymptote parallel to the baſe, we ſhall not inſiſt on this rule further. FIG. 110.

300. Let KO parallel to the baſe be now an aſymptote of the curve FN*e*, and the rectangle contained by PM and DG be always equal to the area APNF, as in prop. 20. Let the rectangle KR be equal to the limit to which the area FNIK continually approaches while the curve FN and aſymptote KI are produced, if any ſuch limit can be aſſigned; and let AR be taken from A towards D when the curve FN*e* is betwixt the baſe AD and aſymptote KI, but in a contrary direction when the aſymptote KI is betwixt the baſe and curve FN*e*. From R towards D take R*d* equal to DG; and, *dh* being parallel to PN on the ſame ſide of the baſe with PN, and equal to AK, join R*h* and it ſhall be the aſymptote of the curve AME. For, if PN meet R*h* in S, PS ſhall be to *dh* (or PI) as RP is to R*d* or DG, and the rectangle contained by PS and DG equal to the rectangle RI. Therefore the rectangle contained by MS and DG is equal to the difference betwixt the area APNF and the rectangle RI, or to the difference betwixt the rectangle RK and the area FNIK. But this difference decreaſes continually while the figure is produced, and may become leſs than any given ſpace, by the ſuppoſition. Therefore MS may become leſs than any given right line; and RS is an aſymptote of the curve AME. But if the area FNIK may exceed any given ſpace by producing the curve and aſymptote, then AME is not a branch of the hyperbolic kind, and no right line can be aſſigned for its aſymptote. The angle however which the tangent at M forms with the ordinate PM approaches continually to the angle FIG. 111.

KA*k*, K*k* being taken upon KI equal to DG. And we are not to conclude that the curve has always an aſymptote, when, according to the language of thoſe who employ infinites and infinitesimals in this doctrine, the curve has an infinite branch that at its termination becomes oblique to the baſe.

301. It is manifeſt, converſely, that if RS be an aſymptote of the curve AME, PN be to the given right line DG as the fluxion of PM is to the fluxion of AP; and, R*d* equal to DG being taken from R in the ſame direction as P is from A, if *dh* parallel to PM meet RS in *h*, then a right line through *h* parallel to the baſe ſhall be the aſymptote of the curve FN*n*; and the rectangle RK ſhall be the limit of the area FKIN, being always greater than this area by an exceſs that decreaſes continually and becomes leſs than any given ſpace by producing the figure.

302. The continuation of thoſe theorems will appear from the following propoſition and its converſe.

PROP. XXIV.

Let the line Bm *be an aſymptote of any kind of the curve* FM; *the fluxion of the baſe being repreſented by the given right line* DG, *let the fluxions of the ordinates* PM, P*m* *be always meaſured by the right lines* PN, P*n*, *and, when* PM *is equal to* AF, *let* P*m*, PN *and* P*n* *be equal to* AB, A*f* *and* A*b*, *reſpectively: Then the rectangle contained by* BF *and* DG *ſhall be the limit of the area bf* N*n*.

FIG. 112.

The curve Bm is an aſymptote of the curve FM (art. 286.) when M*m* continually decreaſes ſo that it becomes equal to any right line how ſmall ſoever that may be given by producing the figure but never vaniſhes. Let FK, BL, *fk* and *bl* parallel to the baſe meet PM in K, L, *k* and *l*, reſpectively. Becauſe PN is to DG as the fluxion of the ordinate PM is to the fluxion of the baſe AP, it follows, that the fluxion of the area APN*f* is equal to the fluxion of the rectangle contained by DG and PM.

There-

Therefore the area APN*f* is equal to the rectangle contained by DG and KM, by theor. 4. ſince AP and KM begin to be generated at the ſame time. In the ſame manner, the area AP*nb* is equal to the rectangle contained by DG and L*m*. And, conſequently, the area *bf*N*n* (the difference of AP*nb* and APN*f*) is equal to the rectangle contained by DG and the difference betwixt L*m* and KM or the exceſs of BF above M*m*. Therefore the area *bf*N*n* is always leſs than the rectangle contained by DG and BF; but it continually approaches to this rectangle as its limit, ſince M*m* continually decreaſes and becomes leſs than any given right line by producing the figure. We have ſuppoſed the lines B*m* and FM to be on the ſame ſide of AP; but the demonſtration is eaſily adapted to the other caſes. It is obvious however that this propoſition cannot be extended to the caſe when the point *m* is found in a rectilineal aſymptote of the curve FM that is perpendicular to the baſe; for AF and PM never meet this aſymptote, being parallel to it.

303. Cor. I. Since M*m* the difference of PM and P*m* continually decreaſes and becomes leſs than any given right line by producing the figure, but never vaniſhes, it follows, that N*n* which meaſures the difference of their fluxions continually decreaſes, and may become leſs than the given right line DG which meaſures the invariable fluxion of the baſe in any given ratio: And therefore *bn* is an aſymptote of the curve *f*N.

304. Cor. II. The converſe of this propoſition eaſily appears: That if *bn* be an aſymptote of the curve *f*N of any kind, and, the point F being taken any where upon A*f* the perpendicular to the baſe at A, BF be taken from F the contrary way that *bf* is from *f*, ſo that the rectangle contained by DG and BF be the limit of the area *bf*N*n*; and if the rectangle contained by KM and DG be always equal to the area APN*f*, and the rectangle contained by L*m* and DG equal to the area AP*nb*: then the curves B*m* and FM ſhall be aſymptotes to each other mutually. For the rectangle contained by M*m* and DG ſhall be equal to the exceſs of the rectangle contained by BF and DG above the area *bf*N*n*; and, ſince this exceſs may become leſs than any given ſpace by continuing the figure, it follows, that M*m* may become leſs than any given right line.

But

Bm if the area *bf*N*n* may exceed any given ſpace by producing the figure, then *Bm* is not an aſymptote of the curve FM.

305. Cor. III. When *Bm* is a right line parallel to the baſe, P*n* vaniſhes, and *bn* coincides with the baſe AP, which therefore is the aſymptote of *f*N, as in art. 293. When *Bm* is a right line oblique to the baſe, P*n* is invariable, and *bn* is a right
Fig. 113. line parallel to the baſe, as in art. 301. When *Bm* is a common parabola that has its axis perpendicular to the baſe, *bn* is a right line oblique to the baſe, and is the aſymptote of the curve *f*N: And if the right line *bn* be given in poſition, and the limit of the area *bf*N*n* be known, the parabolic aſymptote *Bm* is determined by taking BF ſo that the rectangle contained by it and DG may be equal to that limit, producing *nb* till it meet the baſe in R, and upon RI parallel and equal to AB taking IE from I the ſame way as A is from *b*, ſo that the rectangle contained by IE and DG may be equal to the triangle AR*b*; for E ſhall be the vertex of the parabola required: and if RV be taken upon RI equal to 2DG and V*b* parallel to the baſe meet R*b* in *b*, then V*b* ſhall be the parameter of the axis of the parabola.

306. Cor. IV. When *Bm* is a cubic parabola, *bn* is a common parabola; and when *Bm* is ſuch a parabola that the ordinate P*m* is as a power of the abſciſs whoſe index is *m*, then P*n* is as a power of the abſciſs whoſe index is leſs than *m* by unit; and when *m* is leſs than unit, *bn* is ſome hyperbola that has the baſe for its aſymptote. But we have inſiſted on this ſubject ſufficiently.

307. As a right line, or area, may increaſe continually and never amount to a given right line, or rectangle; ſo a ſolid may increaſe continually and never amount to a given cube or cylin-
Fig. 114. der. Let KI parallel to the baſe AP be the aſymptote of any curve *a*ME, as in art. 293. and, K*d* being taken upon KI equal to any given right line DG, let *d*Z parallel to KA meet the baſe in Z and M*x* parallel to the baſe in *g*, and complete the rectangle *augx*. Let the ſquare of PN (the ordinate from the curve FN*e*) be always to the ſquare of DG as the fluxion of PM (the ordinate of the curve *a*ME) is to the fluxion of the baſe AP. Then the ſolid generated by the hyperbolic area APNF

Pl. XII Pa. 234.

Fig. 107.

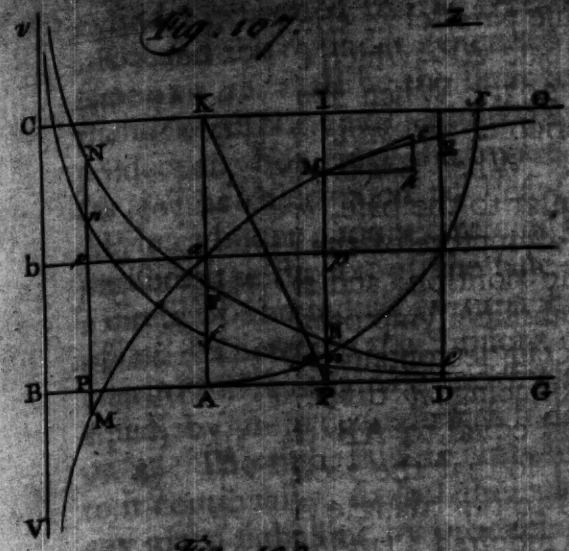


Fig. 108.

Fig. 109.

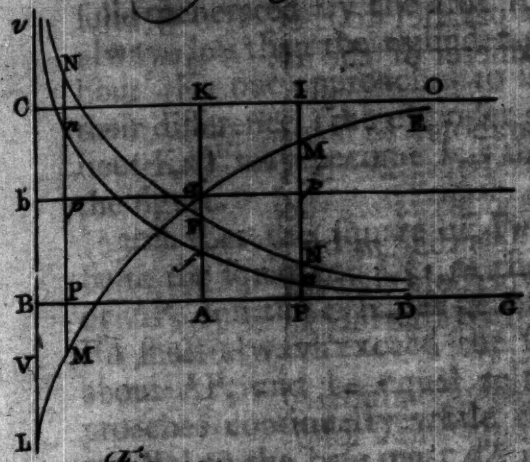


Fig. 110.

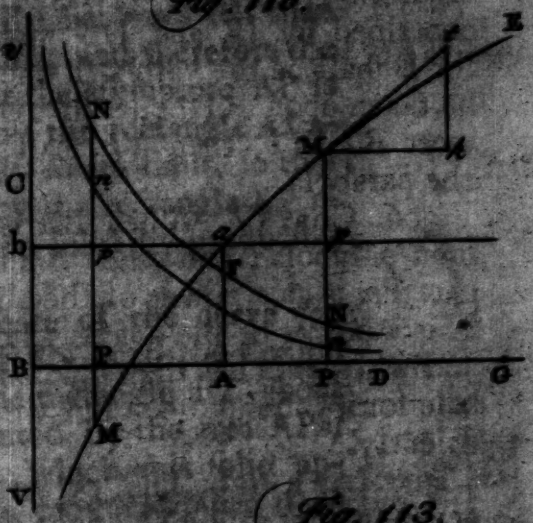


Fig. 111.

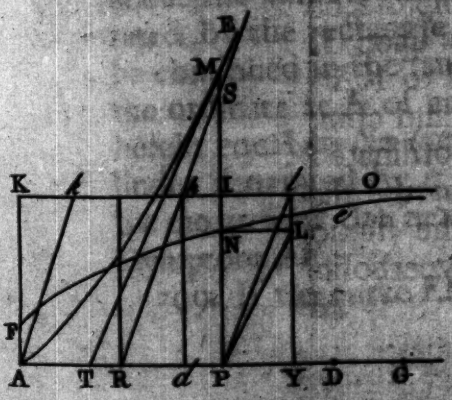


Fig. 112.

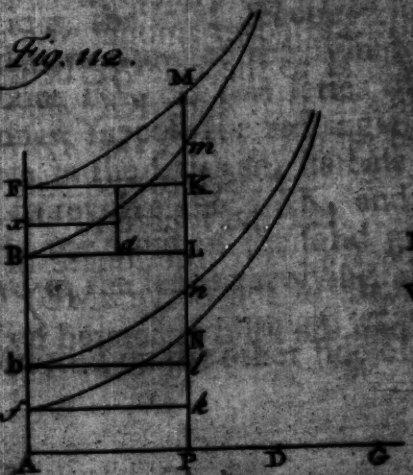
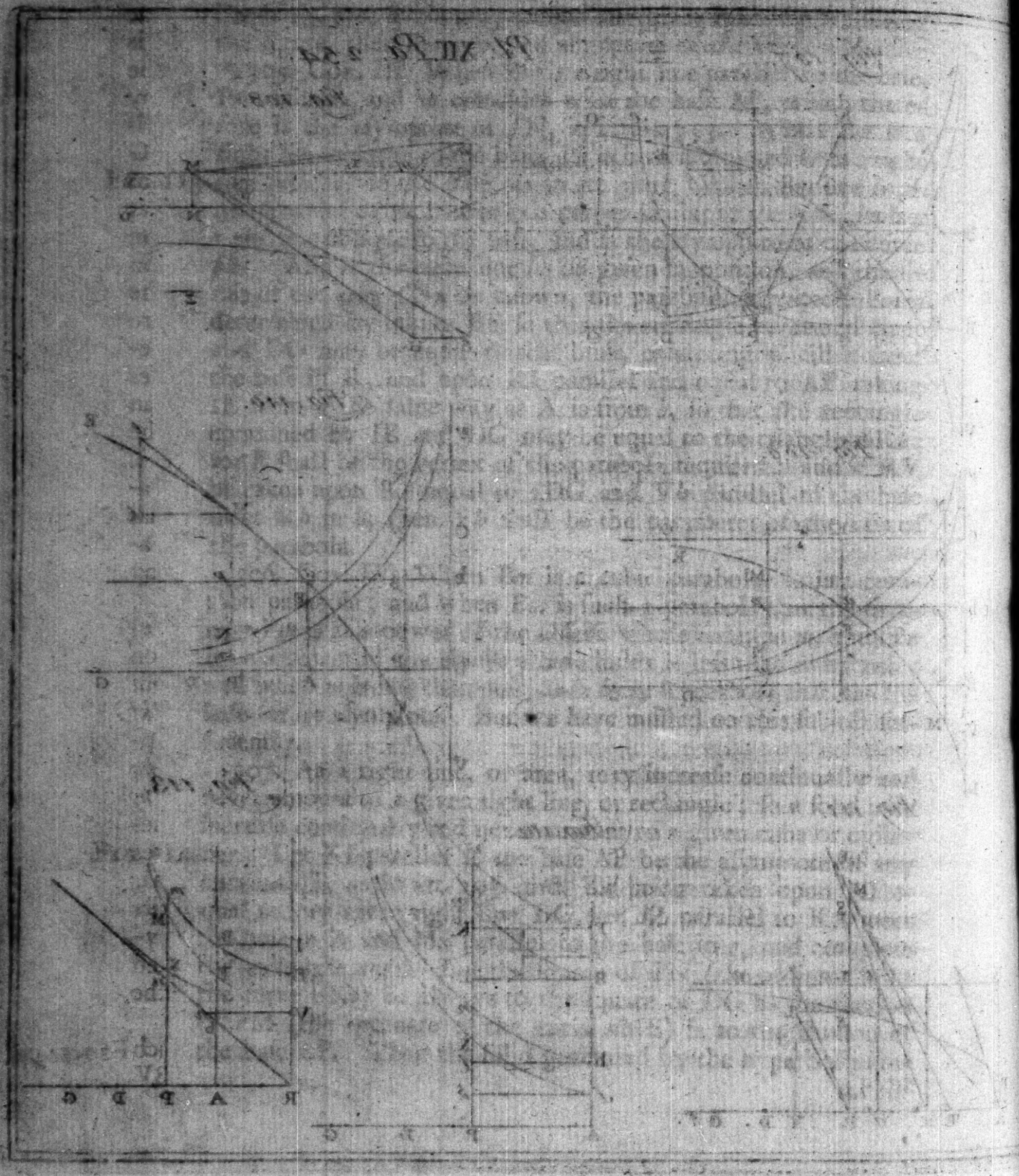


Fig. 113.



APNF revolving about the aſymptote AP ſhall be equal to the cylinder generated by the rectangle *ag* revolving about the axis *a*K; and the cylinder generated by the rectangle *ad* about the ſame axis *a*K ſhall be the limit to which the ſolid generated by the area APNF ſhall continually approach while the figure is produced. For, ſince the ſquare of PN is to the ſquare of DG (or *gx*) as the fluxion of PM (or of *ax*) is to the fluxion of AP, a cylinder upon a circle of the radius PN of a height that meaſures the fluxion of the baſe AP is equal to a cylinder upon a circle of the radius *gx* of a height that meaſures the fluxion of *ax*. Therefore the fluxions of the ſolids generated by the area APNF about AP, and by the rectangle *ag* about *a*K, are equal, by prop. 6. and theſe ſolids themſelves are equal by theor. 4. The right line *ax* is always leſs than *a*K, but approaches to it continually, ſo that their difference IM becomes leſs than any given right line by producing the figure; and therefore the ſolid generated by the area APNF about the aſymptote AP is always leſs than the cylinder generated by the rectangle *ad* about *a*K, but approaches to it continually as its limit, ſo that their difference (the cylinder generated by the rectangle K*g* about K*x*), may become leſs than any given ſolid by producing the figure.

308. Let the ſquare of P*n* (the ordinate of the curve *fn*) be to the ſquare of DG as the fluxion of PN is to the fluxion of AP, and the cylinder generated by the rectangle FZ about FA ſhall always exceed the ſolid generated by the area AP*nf* about AP, and be equal to the limit to which this ſolid approaches continually while the figure is produced; for if *nq* parallel to the baſe meet *d*Z in *q*, the ſolid generated by AP*nf* about the axis AP ſhall be always equal to the cylinder generated by the rectangle F*q* about FA. If this ſeries of curves be continued in the ſame manner, ſuppoſe the rectangle that has the ordinate at A of any curve in the ſeries for its baſe and its height equal to A*z* (or DG) to revolve about AK, and the cylinder generated by it ſhall be the limit of the ſolid generated by the area of the next curve in the ſeries revolving about the aſymptote AP.

309. If the curve EM*a* produced on the other ſide of *a* touch FIG. 115.

BV in L, let LR parallel to AP meet KA and *d*Z produced in *k* and R, and the ſolid generated by APNF about AP when P is betwixt A and B ſhall be always leſs than the cylinder generated
FIG. 114. by the rectangle *a*R revolving about the axis *ak*. But if BV is an aſymptote of the curve EM*a*, the ſolid generated by APNF in this caſe may exceed any given ſolid by producing the figure.

310. When *a*ME is a common hyperbola, CO and CV its aſymptotes, the curve FN*e* is alſo a common hyperbola, and BA and BC its aſymptotes. For in this caſe the fluxion of PM is to the fluxion of AP (or the fluxion of IM to the fluxion of CI) as IM is to CI, or as the rectangle CK*a* to the ſquare of CI; and, by the ſuppoſition, the ſquare of PN is to the ſquare of DG as the fluxion of PM is to the fluxion of AP: Therefore PN is to DG as a mean proportional betwixt CK and K*a* is to BP. From which it follows, that the ſolid generated by the hyperbolic area APNF about AP is always equal to the cylinder generated by the rectangle *ag* about the axis *a*K, and never amounts to the cylinder generated by the rectangle *ad* about *a*K, or BF about BA, but approaches to it as its limit. But if AP be taken from A towards B, the ſolid generated by APNF may exceed any given ſolid by producing the figure. It appears in the ſame manner, that if PN be reciprocally as any power of BP whoſe index is any number that exceeds $\frac{1}{2}$, then a cylinder may be aſſigned that always exceeds the ſolid generated by APNF about the axis AP, the figure being produced
FIG. 116. to any diſtance how great ſoever. When *a*ME is the logarithmic curve and KO its aſymptote, FN*e* is alſo a logarithmic and AP its aſymptote. For, ſince the ſquare of PN is to the ſquare of DG as the fluxion of PM is to the fluxion of AP, or as IM to *a*K, (ſuppoſing *a*K to be the ordinate whoſe logarithm vaniſhes and is equal to the invariable ſubtangent of the curve;) it follows, that the fluxion of PN is to the fluxion of IM as PN is to 2IM. Therefore the fluxion of PN is to the fluxion of AP as PN is to 2*a*K. Hence the cylinder that is the limit of the ſolid generated by the area APNF in this caſe has its axis *a*K equal to one half of AT the ſubtangent, and the radius of the baſe K*d* equal to the ordinate AF. The continuation of the theorems of this kind will appear from the following propoſition.

PROP.

PROP. XXV.

311. *The reſt remaining as in the laſt propoſition, ſuppoſe now that the ſquare of* PN *is to the ſquare of* DG *as the fluxion of* PM *is to the fluxion of* AP, *and the ſquare of* P*n* *to the ſquare of* DG *as the fluxion of* P*m* *is to the fluxion of* AP : *Then, if the area* *bf*N*n* *be ſuppoſed to revolve about the axis* AP, *the ſolid generated by it ſhall be always leſs than a cylinder upon a circle deſcribed with the radius* DG *of an altitude equal to* BF, *but it ſhall approach continually to this cylinder, while the figure is produced, as its limit.* FIG. 112.

It is demonſtrated, in the ſame manner as in art. 307. that if B*d* parallel to AP be equal to DG, and F*x* be taken equal to M*m* from F towards B, the cylinder generated by the rectangle *dx* revolving about B*x* ſhall be always equal to the ſolid generated by the area *bf*N*n* about the axis AP. Therefore the cylinder generated by the rectangle F*d* about the axis FB is the limit of the ſolid that is generated by the area *bf*N*n* about AP.

312. In art. 301. let MT touch the curve AM in M, and meet the baſe in T; and, while the curve AM is produced, the point T ſhall approach continually to R, the angle PTM to the angle PRS, and conſequently the tangent MT ſhall approach continually in poſition to the aſymptote RS. For, let PY be equal to the given right line DG, and Y*l* parallel to PM meet KI in *l*, and NL parallel to the baſe in L; and MT the tangent of the curve at M ſhall be parallel to PL, by art. 247. The right line P*l* is parallel to the aſymptote RS, by art. 301. Therefore the angle LP*l* is equal to the angle formed by the tangent MT and aſymptote RS: And, ſince L*l* (or IN) may become leſs than any given right line by producing the figure while PY remains invariable, ſo that the angle LP*l* may become leſs than any given rectilineal angle; it follows, that MT approaches continually to the aſymptote RS in poſition. Hence, when FIG. 111.

the curve is ſuppoſed to be infinitely produced, the tangent at its infinitely diſtant termination is ſaid to coincide with the aſymptote; which therefore is commonly conſidered as the tangent of the curve at an infinite diſtance. In the ſame manner, when two curves are aſymptotes to each other mutually, they are ſaid to touch each other at an infinite diſtance; and, converſely, when both are ſuppoſed to be infinitely produced, if they are ſuch as muſt be ſuppoſed to meet at their termination, they are ſaid to be aſymptotes of each other.

313. Hence the theorems for drawing tangents to curves often lead to ſuch as ſerve for determining their aſymptotes, there being no more required than to find the right line to which the tangent approaches in poſition while the curve is produced continually. Of theſe we ſhall ſubjoin one, that follows from prop. 18. by which the aſymptotes are determined when a curve is ſuppoſed to be deſcribed by the interſection of right lines that revolve about given points, and the ratio of the velocities with which thoſe lines revolve can be found.

PROP. XXVI.

The reſt remaining as in prop. 18. *let the fluxion of the angle* ACP *be to the fluxion of a* SP (*or the angular velocity of* CP *about* C *to the angular velocity of* SP *about* S) *while* CP *and* SP *become parallel, as* SQ *is to* CQ; *and, if* RX *the aſymptote of the curve deſcribed by* P *meet* CS *in* R, CR *ſhall be equal to* SQ.

FIG. 117. & 118.

It follows from prop. 18. that if PD be the tangent of the curve deſcribed by P, the angle SPT be made equal to CPD according to that propoſition, and PT meet CS in T; the fluxion of the angle ACP ſhall be to the fluxion of the angle *a*SP (or the angular velocity of CP to the angular velocity of SP) as ST is to CT. Let the angle PCV be always equal to PST, and PD meet SC and CV in D and V; let C*x* and S*y* be parallel to the aſymptote RX, and, the angle *x*C*v* being

being made equal to XRC, let C*v* meet RX in *v*. Becauſe the triangles PCV, PST are ſimilar, CV is to ST as CP is to SP; and the ratio of CV to ST approaches continually to a ratio of equality while P deſcribes the branch of the curve that belongs to the aſymptote RX, becauſe while the right lines CP, SP approach to parallelism, their ratio approaches to a ratio of equality: But C*v* is the limit to which CV approaches at the ſame time, becauſe (by the laſt article) the tangent PD approaches in poſition to the aſymptote RX, CP to C*x*, and the angle PCV (or PST) to *y*ST, or XRC, or *x*C*v*. The angle C*v*R being equal to CR*v*, C*v* is equal to CR; and therefore CR is equal to the limit to which ST continually approaches while P deſcribes the curve FH. But ſince the ratio of the fluxion of the angle ACP to the fluxion of *a*SP when CP and SP become parallel, is that of SQ to CQ, by the ſuppoſition; it follows, that SQ is the limit to which ST approaches continually while P deſcribes FH. Therefore CR is equal to SQ. If the angles PCS, PSC increaſe or decreaſe together while CP and SP become parallel, the point Q muſt be taken upon CS betwixt C and S; but it is to be taken upon CS produced beyond C or beyond S, when one of thoſe angles decreaſes while the other increaſes, according as the velocity of SP about S is greater or leſs than that of CP about C: And it is manifeſt, that CR and SQ muſt be always taken in contrary directions from C and S. If one of the angles PCS, PSC decreaſe and the other increaſe while the point P deſcribes FH, and the fluxions of thoſe angles be equal when CP and SP become parallel, the point Q is not then aſſignable, and is ſaid to become infinitely diſtant; in which caſe the branch of the curve deſcribed by P is not of the hyperbolic kind, unleſs ſometimes when CP coincides with CS. FIG. 118. FIG. 117.

314. COR. Let SQ be to CQ as the angular velocity of CP is to the angular velocity of SP at the term or moment when thoſe lines become parallel, and let CR be taken equal to SQ with the precautions we have deſcribed; then RX drawn parallel to CP or SP (which are ſuppoſed parallel to each other) ſhall be an aſymptote of the curve deſcribed by P. If the right lines CP and SP be always tangents of any curve lines that paſs through C and S, inſtead of revolving about thoſe

points, the conſtruction by which the tangent or aſymptote of the curve deſcribed by P is determined, is the ſame. The uſe of the 18th propoſition for drawing the tangents of curve lines, and of this propoſition for determining their aſymptotes, will appear from the following examples, which we chuſe from a great number that might be brought.

315. *Ex.* 1. The right lines CA, S*a* being given in poſition, let the angle ACP be always to *a*SP in any given ratio; determine the point T ſo that ST may be to CT in the ſame given ratio, and that T may lie in the ſame or in an oppoſite direction from C and S, according as the right lines CP, SP revolve about C and S with the ſame or with oppoſite directions; join PT, and make the angle CPN equal to SPT, ſo that thoſe angles may be on oppoſite ſides of CP and SP: And PN ſhall be the tangent of the curve deſcribed by P. The point T is a given or invariable point, becauſe the ratio of ACP to *a*SP is ſuppoſed to be given; let this ratio be that of *m* to *n*, and ST ſhall be to the given line CS as *m* is to the difference of *m* and *n* when CP and SP revolve about C and S in the ſame direction, but as *m* is to the ſum of *m* and *n* when they revolve in oppoſite directions. Let CR be taken from C equal to ST in an oppoſite direction, and the aſymptotes of the curve (if it has any, that is, if CP and SP ever become parallel while the curve
FIG. 119. is deſcribed) ſhall paſs through R. To mention a few of the ſimpleſt caſes that are comprehended in this example, let ACP be equal to *a*SP, and if CP and SP revolve about C and S with the ſame direction, the point P ſhall deſcribe a circle: In this caſe PT is to be conſidered as parallel to CS, and the angle CPN being made equal to SPT or PSC, PN is the tangent of
FIG. 120. the circle, as is well known. But if CP and SP revolve about and S with oppoſite directions, the point P ſhall deſcribe an equilateral hyperbola. For if S*a* be a right line given in poſition, ASP be always equal to SCP, PH parallel to SA meet CS in H; the triangles HPS, HPC being ſimilar, the rectangle SHC ſhall be equal to the ſquare of PH, and P ſhall deſcribe an equilateral hyperbola, of which CS is a diameter and SA a tangent at S. Biſect CS in T, join PT, let the angle SPN be made equal to CPT, and PN ſhall be the tangent of the hyperbola

perbola at P: As in the circle, if SPN be made equal to CP*i*, *i* being the center and CS a diameter, PN is a tangent; only the angle SPN in the circle is taken the contrary way. Some other analogies betwixt the circle and equilateral hyperbola appear alſo from this: 1. Let CS any diameter of the hyperbola be drawn and right lines from C and S to P any point in the hyperbola, if the angle PSA be made equal to PCS, SA ſhall be the tangent at S: The tangent of the circle at S is determined by the ſame conſtruction, only that angle is to be taken on the other ſide of SP. 2. When any two points P and *p* are in the equilateral hyperbola, and right lines are inflected from C and S, the extremities of any diameter CS that paſſes through the center T, to thoſe points, the angle PC*p* is equal to PS*p* or to its ſupplement to two right ones. 3. The angle PTS contained by TP, TS any two ſemidiameters of the hyperbola is equal to the angle PVA contained by SA, PN the tangents at the extremities of thoſe ſemidiameters: For produce PT till it meet the oppoſite hyperbola in K, join CK, CS; and, TV which is parallel to CP (becauſe it biſects CS and SP) and the angle PTV being equal to CPT or SPV, and STV equal to TSK or PSV, it follows, that PTS is equal to PVA. As for the aſymptotes, becauſe the angular velocity of CP is equal to the angular velocity of SP, it follows from the laſt propoſition, that they paſs through T, the point R coinciding with T; and, if PB, P*b* be taken equal to PS, the aſymptotes ſhall be parallel to SB and S*b*, becauſe, when S*p* comes into the poſition of SB or S*b*, then C*p* and S*p* become parallel. FIG. 119. & 120.

316. If the angle PSC be always double of PCS, take ST from S towards C equal to one third part of SC; and the point P ſhall deſcribe an hyperbola whoſe tranſverſe axis ſhall be CT, and that ſhall have one of its *foci* in S: And, if the angle CPN be made equal to SPT, according to the laſt propoſition, PN ſhall be a tangent. Take CR equal to ST, or one third part of CS, and the aſymptotes ſhall paſs through R conſtituting with RS an angle of 60 degrees, becauſe CP and SP become parallel when the angle PCS is of that magnitude. If the angle PCB be always double of PSB, (the point B being ſituated on CS produced beyond C,) it is obvious, that P ſhall deſcribe FIG. 121. FIG. 122.

ſcribe a circle paſſing through S having its center in C, and that T is the extremity of the diameter through S. If CA and S*a* form any given angles with CS, ACP be always double of *a*SP, and the angular motions of CP and SP be in the ſame direction about the points C and S, the point P ſhall deſcribe a line of the third order that has a double point in S, of the 34th kind according to Sir ISAAC NEWTON's enumeration: And if CT be taken upon SC produced beyond C equal to CS, and the angle SPN be made equal to CPT according to the laſt propoſition, PN ſhall be the tangent. If CR be taken from C equal to ST but in a contrary direction, the aſymptote ſhall paſs through R; and, S*d* being parallel to CA, if the angle *a*SD be made equal to *a*S*d* on the oppoſite ſide of S*a*, the aſymptote ſhall be parallel to SD. If CP meet the right line SD given in poſition in M, MP ſhall be always equal to MS; and therefore this curve coincides with one, of which ſeveral properties are demonſtrated elſewhere, *(lemma 2. par. 1. Deſcrip. curvar.)* If the angular motion of SP about S be triple of the angular motion of CP about C, and be in the ſame direction, ſo that TSP be always triple of TCP, the point P ſhall deſcribe a line of the third order of the ſame *ſpecies* with the former, only in this curve CT is triple of CR. If AM perpendicular to CS biſect it in A, and CP meet AM in M, SP ſhall be equal to SM or CM; becauſe the angles SMP, SPM are equal, being each double of SCP: Therefore, ſince the angle TSP is triple of TCP, the area TSP is triple of the triangle ACM; and if ACK be an angle of 60 degrees, the area bounded by the curve CPT and right line CT ſhall be triple of the triangle CAK. The limit of the area included betwixt CR, the aſymptote RX and the curve, is alſo triple of the ſame triangle. Let C*p*T be the curve called the *foliat*, that has the ſame aſymptote RX, the ſame double point in C, and paſſes through T; let PV perpendicular to RT meet the foliat in *p*, then PV ſhall be to *p*V in the invariable ratio of AK to CA, or of the ſquare root of 3 to unit. And hence the quadrature of the foliat and its tangents may be determined: For the area S*p*T muſt be to to SPT (or 3CAM) as 1 is to $\sqrt{3}$; and, the angle CPN being made equal to SPT, if NP meet RT in *n*, the right lines P*n*,

FIG. 123.

FIG. 124.

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Fig. 114.

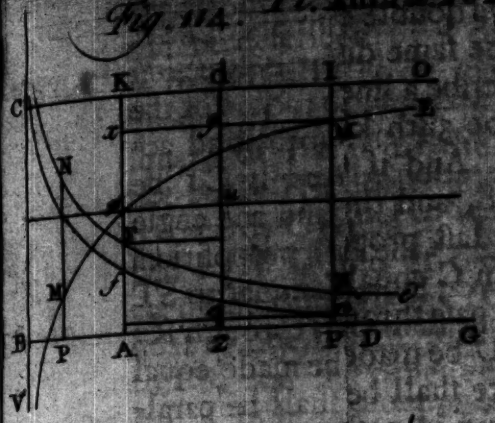


Fig. 115.

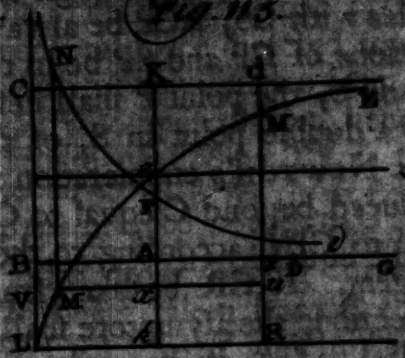


Fig. 116.

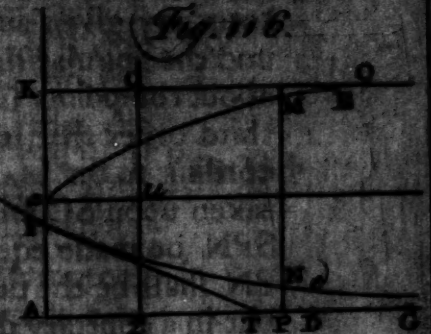


Fig. 117.

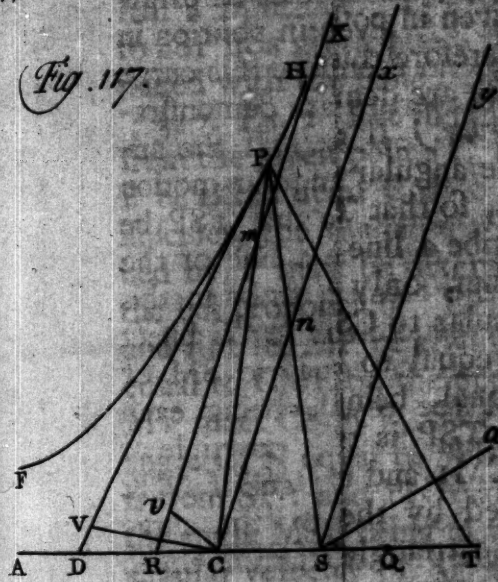


Fig. 118.

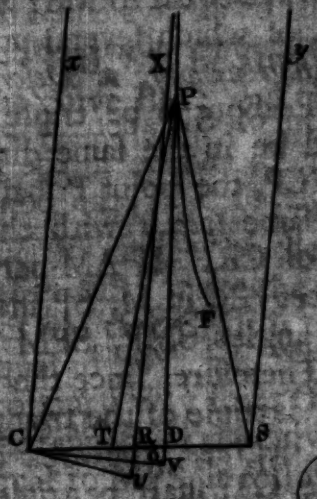


Fig. 121.

Fig. 122.

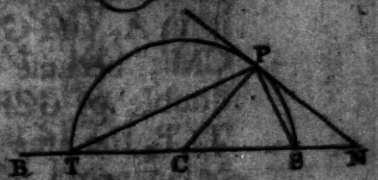


Fig. 120.

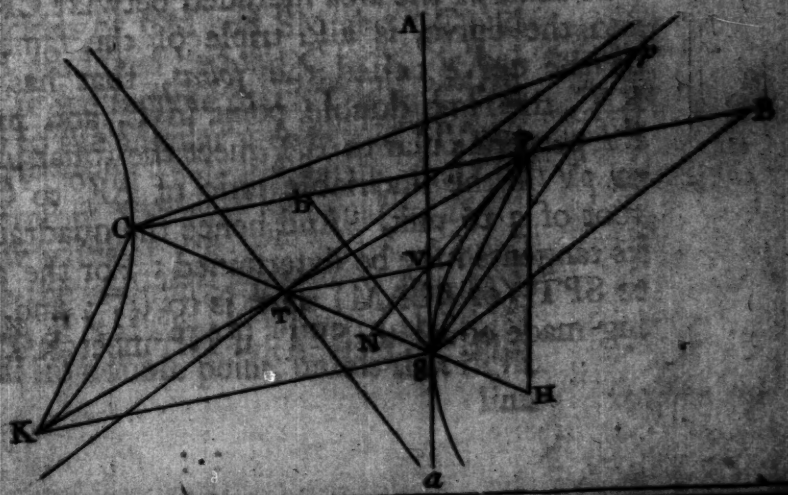
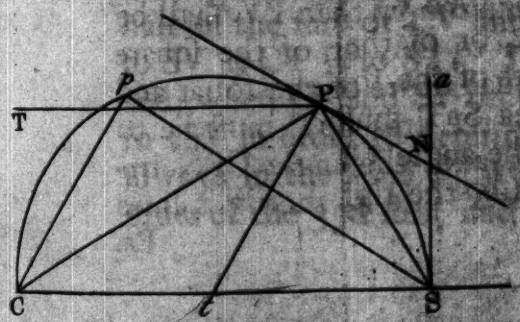
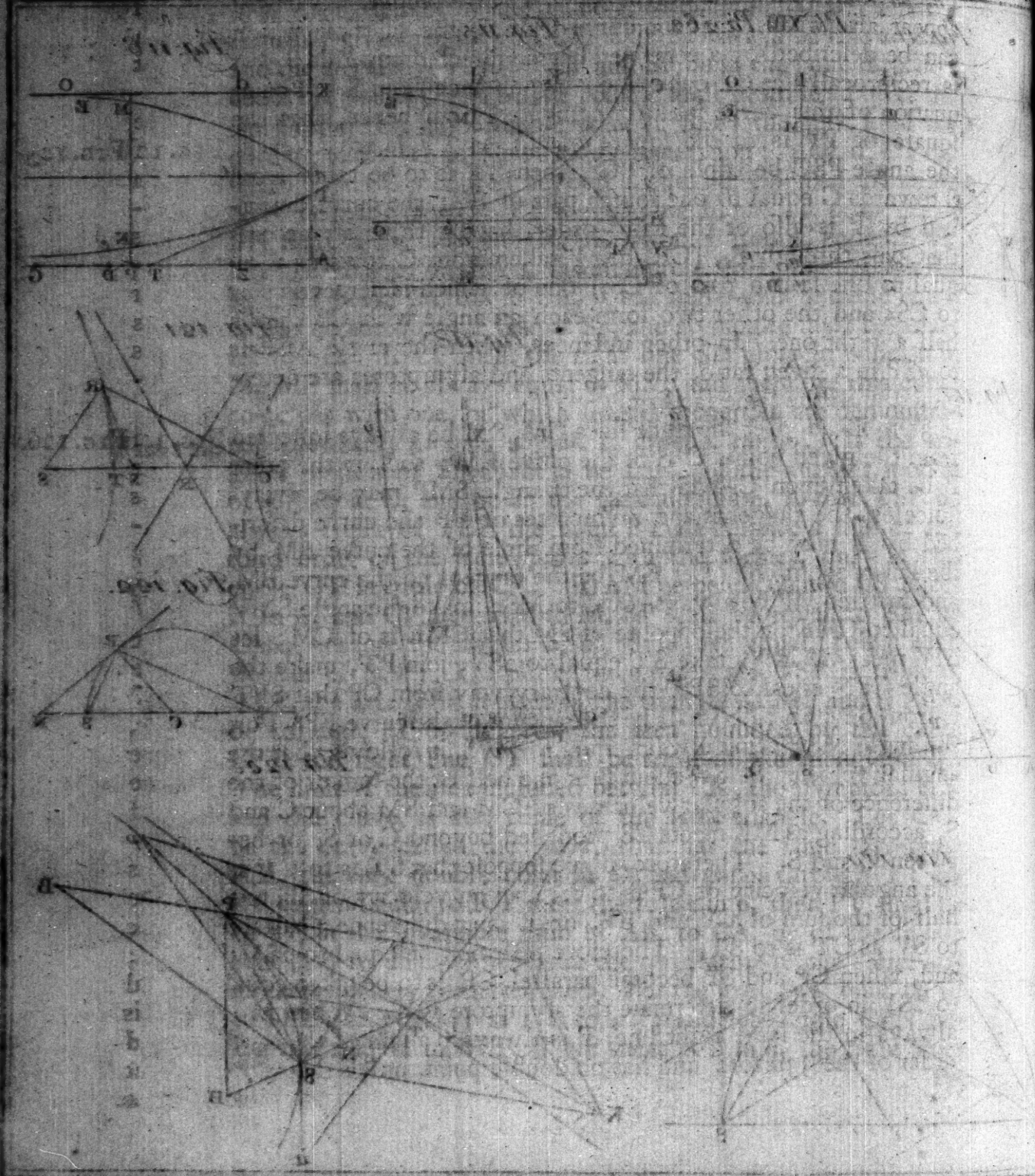


Fig. 119.





Pn, *pn* ſhall touch the curves TPC, T*p*C reſpectively. It will appear afterwards, that the curve TPC is one of thoſe which can be deſcribed by a centripetal force directed towards S that is reciprocally as the cube of SP the diſtance from S. The equation of this curve is eaſily computed from hence, that the ſquare of PV is to the ſquare of CV as TV is to RV. If the angle PSC be triple of PCS, then ST is to be taken from S towards C equal to one fourth part of SC; the curve deſcribed by P is alſo of the third order, having three aſymptotes that paſs through R, (CR being taken from C towards S equal to one fourth part of CS,) one of which is perpendicular to CS, and the other two form each an angle with CS that is half a right one. In other inſtances, when the angle ACP is to *a*SP in a given ratio, the tangents and aſymptotes are determined with the ſame facility. FIG. 125.

317. *Ex.* 2. Let the right lines CM, SM be always inflected from the given points C, S to the curve BM; and, if the point P be taken upon CM ſo that the triangle SMP may be always iſoſceles, the tangents and aſymptotes of FP the curve deſcribed by P may be determined from thoſe of the curve BM by the laſt propoſition. Let EM be the tangent of the curve BM, and firſt let MP be always equal to MS; make the angle SMV equal to CME on the oppoſite of SM that CME is of CM; let MV meet CS in V, take ST equal to 2SV, join PT, make the angle CPN equal to SPT the contrary way from CP that SPT is from SP: and PN ſhall be the tangent of the curve FP. For, the angle MSP being always equal to MPS, it follows, that the angular velocity of SP about S is one half of the ſum or of the difference of the angular velocities of CM and SM about C and S, according as ME meets CS produced beyond C or S, or between C and S. Therefore, if we ſuppoſe that ST is to CT as the angular velocity of CP is to the angular velocity of SP, one half of the ſum of CV and SV, or of their difference, ſhall be to SV as CT is to ST. Therefore ST muſt be equal to 2SV; and, when CP and SP become parallel, CR is to be taken equal to 2SV in order to determine the aſymptote RX. When M is always in the ſame right line, P is always in a line of the third order of the 33d ſort, and has no double point, unleſs that right FIG. 126.

line

line pass through S; in which case the curve FP coincides with one of those which we considered in the last article. But if we suppose that the side SP is always equal to SM, then, the rest of the construction being the same, ST is to be taken equal to SV. Lastly, if PM be always equal to PS, ST is to be taken equal to one half of SV. Of this last case we have an example in
Fig. 127. the ellipse and hyperbola: For when M is always found in a circle described from the center C, P is found in an ellipse or in an hyperbola according as the point S is within or without the circle, C and S are the two *foci*, CM is equal to the transverse axis, and the angle CME is a right one: Therefore, SMV being made a right angle, since SP is equal to PM, if PH bisect SM in H, it shall bisect SV in T, and shall touch the curve FP; because, the angles SPT, CPN being equal, PN and PT coincide. If, instead of a right line revolving about C, we substitute a right line moving parallel to itself, the same constructions may be easily adapted for determining the tangents and asym-
Fig. 128. ptotes of the curve that shall be described by P: And, supposing M to be always found in a right line, we shall have an example of this in the parabola when MP is always equal to PS, and in the equilateral hyperbola when MP is equal to MS.

Fig. 129. 318. *Ex.* 3. Let the angular velocity of CP about C be to the angular velocity of CM about C as any right line C*m* is to CS; and the angular velocity of SP about S to the angular velocity of SM about S as S*n* is to CS. Then, the rest of the construction being the same as formerly, let ST be to CT in the ratio compounded of that of C*m* to S*n* and that of SV to CV. This and the preceeding examples might be easily rendred more general.

Fig. 130. 319. *Ex.* 4. Let the invariable angles DCG, KSH revolve about the given points C and S; let M the intersection of the sides CD, SK describe the curve BM, and let FP be the curve described by P the intersection of the other sides CG, SH. Let AM be the tangent of the curve BM, make the angle SMT equal to CMA, join TP, and make CPN equal to SPT, with the precautions that have been mentioned so often, and PN shall be the tangent of the curve FP. For the angular velocity of CM about C is to the angular velocity of SM about S as ST is to CT, by

prop.

prop. 18. The angular velocities of CP and SP about C and S are reſpectively equal to the angular velocities of CM and SM about the ſame points, becauſe the angles MCP, MSP are invariable. Therefore the angular velocity of CP is to the angular velocity of SP as CT is to ST; and, by the converſe of the 18th propoſition, PN is the tangent of the curve FP. When M comes to *m*, let SP and CP become parallel; and, *ma* being the tangent at *m*, let S*m*Q be made equal to C*ma*; and, if *m*Q meet CS in Q, take CR equal to SQ the contrary way from C that Q is from S, draw RX parallel to CP or SP, and it ſhall be an aſymptote of the curve FP: but if *m*Q be parallel to CS, the branch of the curve deſcribed by P while M comes to *m* ſhall not be of the hyperbolic kind. When the point M deſcribes a right line, P deſcribes a conic ſection, unleſs when CP and SP coincide at the ſame time with CS; in which caſe P likewiſe deſcribes a right line. When M deſcribes a conic ſection that paſſes through one of the poles C, S only, and the right lines CP, SP coincide not with CS at the ſame time, P deſcribes a line of the third order that has a double point in that pole. When the conic ſection paſſes through neither of the poles, P deſcribes a line of the fourth order that has three double points: And by theſe conſtructions the tangents and aſymptotes of thoſe curves, and of ſuch as can be reduced to ſimilar claſſes of the higher orders, are determined. While the invariable angles MCP, MSP revolve about C and S, and M deſcribes a right line, if the given angle P*ſp* revolve about another given point *ſ* and the ſide *ſp* always meet CM in *p*; then ſhall *p* deſcribe a line of the third order that ſhall have a double point in C, becauſe P deſcribes a conic ſection that paſſes through C. The tangent at *p* is determined by drawing P*t*, ſo that the angle *ſPt* be equal to SPT, meeting C*ſ* in *t*, joining *pt* and conſtituting the angle C*pn* equal to *ſpt*: for *pn* ſhall be the tangent; becauſe the angular velocity of *ſp* is equal to that of *ſP*, and is to the angular velocity of CP, or of C*p*, as C*t* is to *ſt*.

320. *Ex.* 5. Let the angle MCP revolve about C as in the preceeding example; but, the angle PQM being invariable, let the angular point Q deſcribe the curve IQ, and the ſide QM always paſs through the pole S; let A*a*, B*b* and PN be tangents FIG. 131.

gents of the curves IQ, GM and FP respectively. Let the angle QSD be made equal to DQ*a*, (so that a circle described through S, Q and D may touch A*a*;) join CD, and let the angles DPT, SMH be made equal to the angles CPN, CMB respectively, with the precaution so often mentioned; join TH, and it shall be parallel to SD: And hence, the tangent of IQ being given with the tangent of either GM or FP, the tangent of the other is easily determined. The asymptotes of the branches that are described by M when CM and SM become parallel, or by P when CP and QP become parallel, are determined by a construction that easily follows from this. When the point N describes a right line, and either of the points M, P describe also a right line, the other point describes a line of the third order (some cases excepted) that has a double point. The demonstration of this construction is easily deduced from prop. 18. For it follows from what was shewn there, that the angular velocity of DQ about D is equal to the angular velocity of SQ about S; so that if A*a*, PN and B*b* are the tangents, CT must be to DT as CH is to SH, and TH parallel to SD.

FIG. 132. 321. *Ex.* 6. The lines of the third order that have a *double* point are also described by P when the right lines CM, DM and SP revolve about the poles C, D and S, so that M describes a conic section that passes through C, and SP is always parallel to DM. Let M*t* touch the conic section, and, the angle DMV being made equal to CM*t*, let MV meet CD in V; draw VH parallel to DS meeting CS in H; and, the angle CPT being made equal to SPH with the usual precaution, then PT shall be the tangent of the curve described by P. One of the asymptotes is parallel to CD, and its position is determined by producing CD till it meet the conic section in E, drawing the tangent of the section EQ meeting CS in Q, and taking CR equal to SQ according to the last proposition. Hence a method is easily derived for drawing a line of the third order through six given points, one of which is to be a double point, that shall have an asymptote parallel to a right line given in position, or a branch whose tangent approaches to this right line as its limit, as those curves always have. When M describes any other curve, the tangents and asymptotes of the curve described by P are deter-

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Fig. 123.

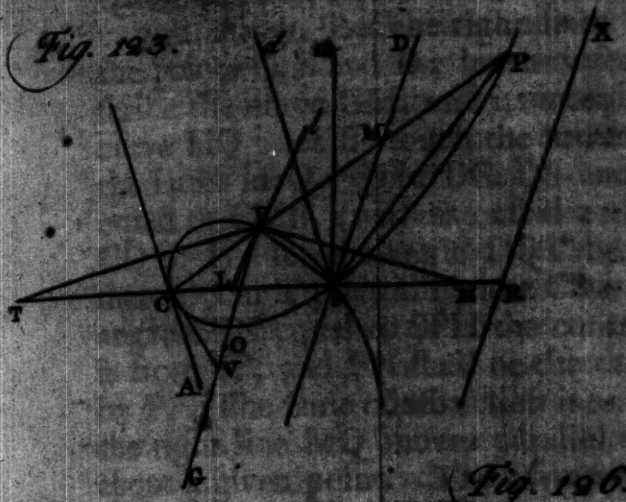


Fig. 124.

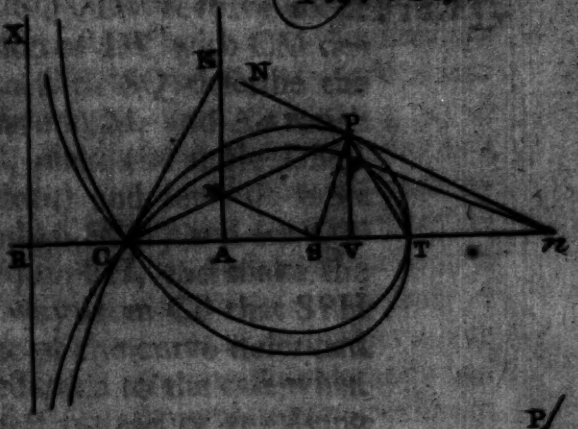


Fig. 125.

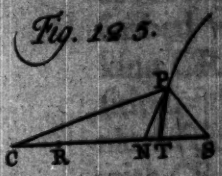


Fig. 126.

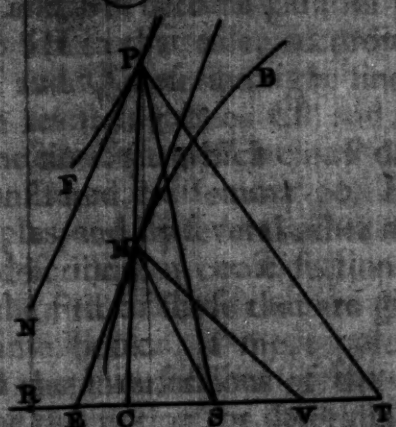


Fig. 127.

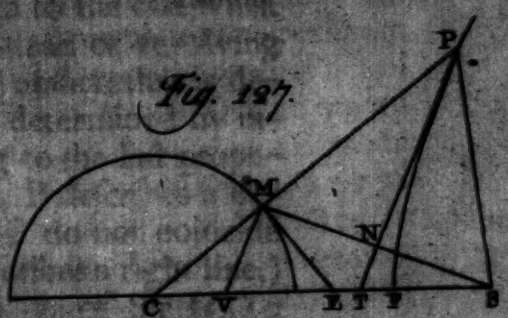


Fig. 129.

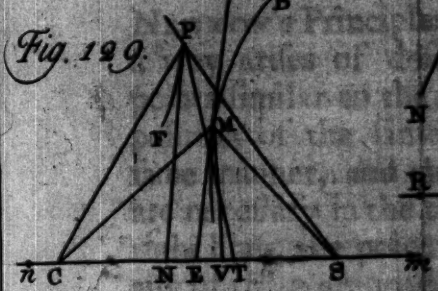


Fig. 128.

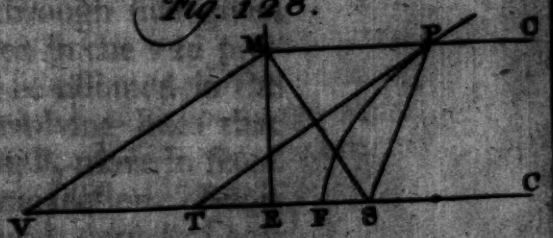


Fig. 130. N. 1

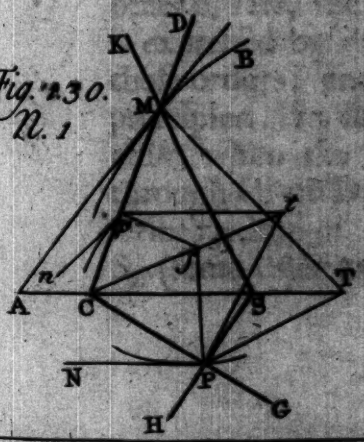


Fig. 130. N. 2

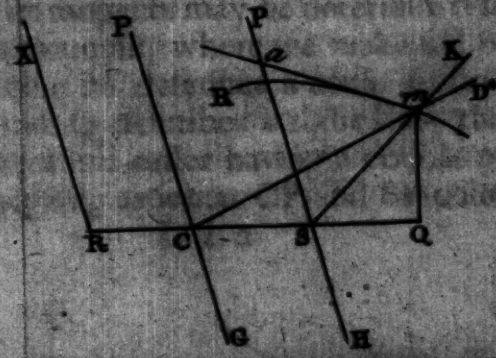
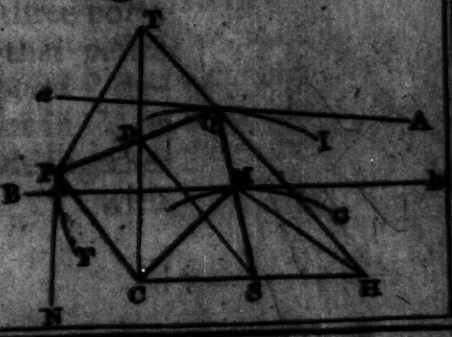
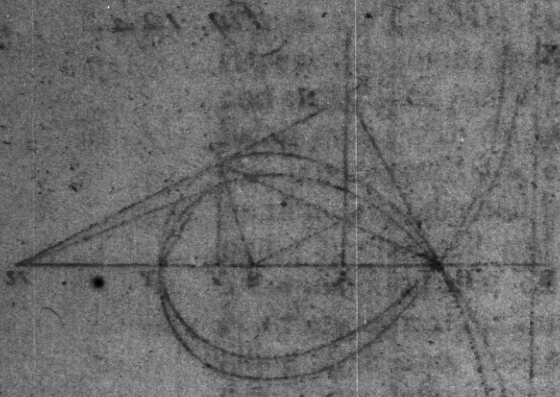
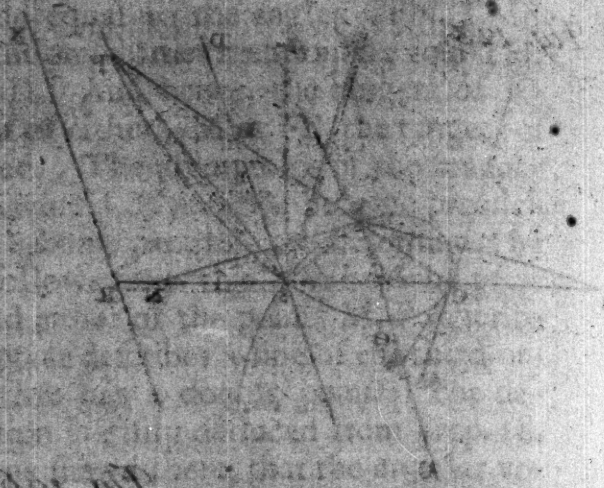
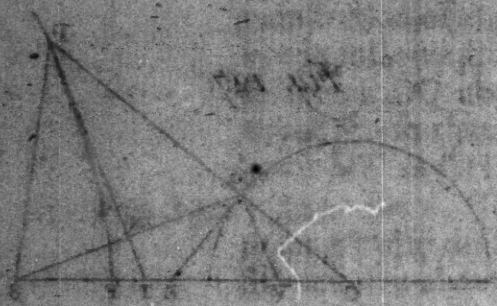


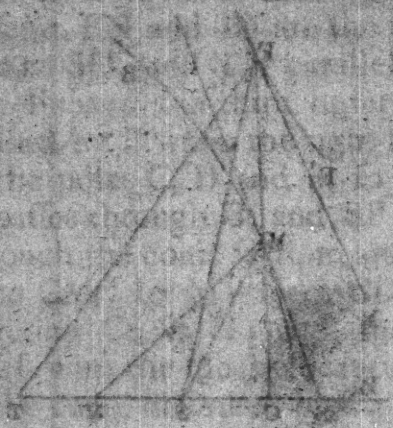
Fig. 131.

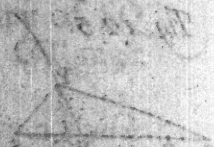


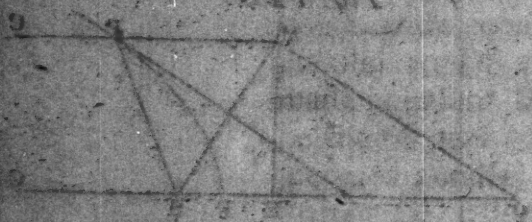


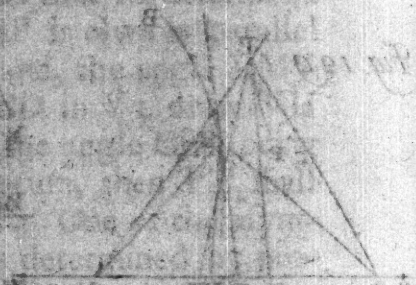


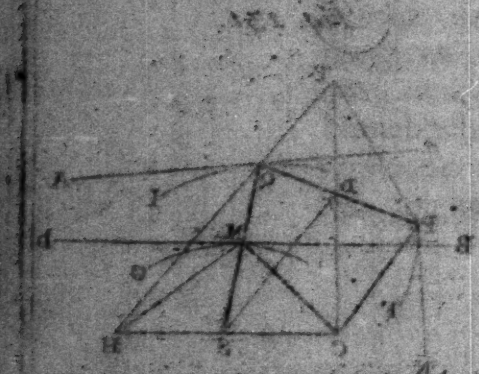


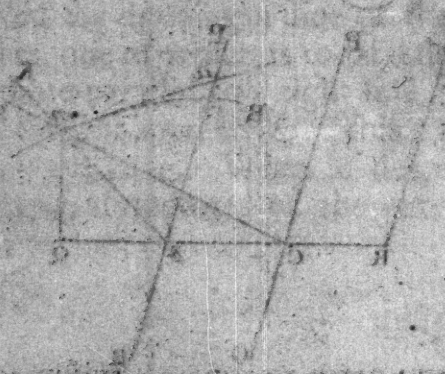


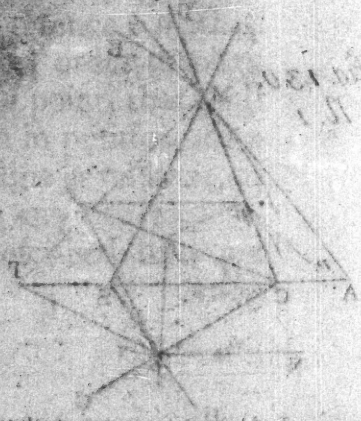












determined in the ſame manner. Thus, in the conchoid, if *m*H parallel to SA meet SH parallel to AD in H, and the angle QMT be made equal to SMH, MT is the tangent. FIG. 108.

322. *Ex.* 7. Let the right lines CM, SQ, DV revolve about the points C, S and D; let the interſection of DV with CM deſcribe the curve GM, and its interſection with SQ deſcribe the curve HQ; let B*b* touch the former curve in M, and A*a* touch the latter in Q. Join DC, DS and CS, make the angle DML equal to CMB, (with the uſual precaution,) and let ML meet CD in L; let the angle DQT be equal to SQA, and let QT meet DS in T; let LT meet CS in H, join PH, and make the angle CPN equal to SPH the contrary way from CP that SPH is from SP, and PN ſhall be the tangent of the curve deſcribed by P. The ſame conſtruction is eaſily adapted to the caſe when the right line MQ moves parallel to itſelf inſtead of revolving about a given point. The aſymptote of the branch that is deſcribed when CP and SP become parallel is determined by taking CR equal to SH in that caſe, according to the laſt propoſition. When M and Q deſcribe right lines, P deſcribes a conic ſection, if the revolving lines CP and SP do not coincide with CS at the ſame time, (in which caſe P deſcribes a right line,) as may be demonſtrated by lemma 20. lib. 1. of Sir ISAAC NEWTON's Principles, and by ſeveral other methods: And hence a way ariſes of deſcribing a conic ſection through five given points ſimilar to the firſt of thoſe that are given in the 22d propoſition of the ſame book. If more poles be aſſumed in the ſame manner, and the interſections of the revolving lines that are neceſſary in the deſcription for determining P, move in fixed right lines, the point P ſhall ſtill deſcribe a conic ſection. But if any of them are made to deſcribe conic ſections or curves of any other orders, the line deſcribed by P may ariſe to the order that is expreſſed by the double product of the numbers that expreſs thoſe orders; and its tangents may be determined from the ſame principles. In this example, where we make uſe of three poles only, when the point M deſcribes a conic ſection that paſſes through C, or D, and Q deſcribes a right line, the point P deſcribes a line of the third order having a double point in C, or S, (ſome caſes excepted, as when CP and SP coincide together FIG. 133.

with CS:) And hence other methods arise for drawing such curves through seven points, one of which is supposed to be double, and for determining its tangents and asymptotes. The conic section GM may be left out in explaining the manner in which the curve is described, or that in which its tangents and asymptotes are found, by substituting the construction by which the points and tangents of GM are determined, in its place; of which substitution we gave an example at the end of art. 319. This may be easily extended to some classes of the higher orders of lines. The lines that are described in this manner have always one or more of those points that are called *double*, or that are more complex: for though, in particular cases, some of those points become simple; yet one or more still remain double, or the
FIG. 134. curve becomes a conic section. Thus, if M describe a conic section that passes neither through C nor D, while Q describes a right line AQ, P describes a line of the fourth order, that has three double points, *viz.* C, S, and the point where DC meets AQ; unless when CP and SP coincide at the same time with CS, in which case the points C and S are no longer double, and the curve described by P is of a lower order; but the curve is then either of the third order and has still a double point, or becomes a conic section: And the lines of the third order, that have no *double* point, cannot be described, when the angles or right lines are all made to revolve about fixed points, or poles, by any method at least published hitherto. When the angular points are carried along right lines, as in the following example, such curves may be comprehended in the description likewise.

FIG. 135. 323. *Ex.* 8. The angles PML, PQL being invariable, let A*a*, B*b* and D*d* be the tangents of the curves described by the points M, Q and L, and let PM always pass through a given point C, and PQ through a given point S; make the angle MCG equal to GM*a* or AML, QSH equal to BQL, (so that a circle through C, M and G may touch A*a*, and a circle through S, Q and H may touch B*b*,) and let CG, SH meet LM and LQ in G and H, respectively; join GH meeting D*d* in I, make the angle GLZ equal to HLI with the precaution we have often mentioned; let CT be to ST as GZ is to HZ, so that the points C, T and S may be in the same situation with respect to each other as G,

L and H; and if the angle SPN be made equal to CPT the contrary way from SP that CPT is from CP, then PN ſhall be the tangent of the curve deſcribed by P. When M, Q and L deſcribe right lines, P deſcribes a line of the fourth order, in which C and S are double points, unleſs the revolving lines CP and SP coincide at the ſame time with CS. But in this laſt caſe (that is, ſuppoſing CS to meet *Aa* and *Bb* in A and B, the angle SBK equal to SQL, and BK to meet *Dd* in K, when CML is equal to CAK) P deſcribes a line of the third order, the points C and S are ſimple, (*Deſcript. curvar. prop.* 15. *par.* 1.) and the curve interſects CS in a third point *n*, the ſame to which T comes when M comes to A and Q to B. And thus lines of the third order are deſcribed that have no *double* point, and their tangents and aſymptotes are determined from the deſcription. The ſame principles ſerve for determining the tangents and aſymptotes of the curves that are generated when more lines, angles and poles are made uſe of in the deſcription.

324. Theſe examples may ſerve to ſhew how theorems relating to the deſcription of lines are of uſe in reſolving problems concerning them; and this will farther appear when we come to treat of the curvature of lines. The tangents at the points C and S in the preceeding examples are determined more eaſily from art. 214. in a manner that may be applied in ſome caſes to any other points of the curve. For example, let A, B, C, S and E be five points in a conic ſection, and let it be required to draw a tangent at C; join BC, SE, and let them meet in D; let SC meet AB in *q*, and *q*D meet AE in *n*; join C*n*, and it ſhall be the tangent of the conic ſection at C. Again, let a line of the third order have a double point in C, and let it be required to draw a tangent to any other point in the curve, as S. Let any right line through S meet the curve in other two points *f* and *d*, and let another right line meet the curve in the three points *a*, *b* and *e*; join C*e* meeting S*d* in D. Let a conic ſection deſcribed through the five points C, *a*, *b*, *d* and *f* meet CS in *g*, join D*g* meeting *ab* in *k*, then join S*k*, and it ſhall be the tangent at S. Or the point *g* may be found without deſcribing the conic ſection, by producing *db* and *a*C till they meet in *l*, *l*S and *af* till they meet in *h*, and joining *hk* which produced (if neceſ-

Fig. 136.

Fig. 137.

ſary)

ſary) ſhall meet CS in *g*. This tangent may be determined by ſeveral other methods from the ſame principles, and ſome properties of ſuch lines may be deduced analogous to thoſe demonſtrated of the conic ſections by the learned Mr. SIMSON, *Sect. conic. lib. 5. prop. 45. &c.* Many other examples might be brought of this kind; but we have inſiſted on this ſubject at a ſufficient length. We have deſcribed theſe examples in this rather than in the ſeventh chapter, that we might conſider the tangents and the aſymptotes together.

325. By an aſſignable magnitude we always underſtand any finite magnitude, and by an aſſignable ratio that of any finite quantities.

PROP. XXVII.

Let Pr, *the ordinate of the figure* APrg, *be to* PN, *the ordinate of the figure* APNF, *in any ratio that approaches to an aſſignable ratio, as its limit, while the figures are produced; let* PN *be reciprocally as any power of* BP, *and if the exponent of this power* FIG. 138. *be greater than unit, and* AP *be taken upon* BA *produced beyond* A, *the areas* APNF, APrg *ſhall both have limits; but if that exponent be not greater than unit, thoſe areas may be produced till they exceed any given ſpace. It is the contrary, when* AP *is taken from* A *towards* B.

FIG. 107. Let IM in art. 293. be reciprocally as any power of CI (or BP) whoſe exponent is any poſitive number or fraction expreſſed by *m*, and the fluxion of IM ſhall be to the fluxion of CI (or of the baſe AP) in the ratio compounded of that of IM to CI and that of *m* to unit, (art. 167. & 168.) Let PN be to DG (as in art. 293.) as the fluxion of PM (or of IM) is to the fluxion of the baſe AP: then PM ſhall be as IM directly and BP inverſely, and therefore inverſely as ſome power of BP whoſe exponent exceeds unit by *m*. When AP is taken upon BA produced beyond A, the area APNF is always leſs than the rectangle contained

tained by *a*K and DG, which is its limit, by art. 293. But if AP be taken from A towards B, the area APNF may be produced till it exceed any given ſpace, and has no ſuch limit, by art. 294. Therefore, converſely, if PN be reciprocally as any power of BP whoſe exponent exceeds unit by any integer number, or by any fraction how ſmall ſoever, then the area APNF has a limit when AP is taken upon BA produced beyond A, but has no limit when it is taken from B towards A. From this it eaſily follows, (and it may be likewiſe ſhewn from art. 295.) that, when PN is reciprocally as any power of the baſe whoſe exponent is leſs than unit, the area APNF has a limit which it never can exceed if AP is taken from A towards B, but has no ſuch limit if AP is taken upon BA produced beyond A. This being premiſed, becauſe the baſe AP is an aſymptote of the figure APNF, and the ratio of P*r* to PN approaches to an aſſignable ratio as its limit while the figure is produced, it follows, that AP is alſo an aſymptote of the figure AP*rg*. Suppoſe that A*g* is an ordinate at ſuch diſtance from A, that when AP is taken upon BA produced beyond A, P*r* never coincides with any aſymptote the figure AP*rg* may have parallel to the ordinates; and ſince the ratio of P*r* to PN is never greater than an aſſignable ratio while AP is any aſſignable diſtance, and the limit to which their ratio approaches (whether by increaſing or decreaſing) while the figure is produced, is alſo aſſignable; it follows, that there may be a ratio of a greater finite quantity to a leſſer which may exceed any ratio of P*r* to PN. Let A*b* be to AF, and the ordinate P*x* always to PN, in ſuch a ratio; and the figure AP*xb* ſhall be always to APNF in the ſame ratio, (by art. 111.) Therefore, if there is a limit which the area APNF never can amount to, there is alſo a limit which the area AP*xb* never can amount to, which is to the former as A*b* is to AF. But P*x* is always greater than P*r*, and the area AP*xb* greater than AP*rg*; therefore there is likewiſe a limit which the area AP*rg* never can amount to. If the area APNF may be produced till it exceed any given ſpace, that is, if it have no limit, then let P*x* be ſuppoſed to be to PN in any invariable ratio leſs than the leaſt ratio of P*r* to PN, and the area AP*xb* ſhall be always leſs than AP*rg*; but ſince it is in an invariable ratio

Fig. 138.

Fig. 139.

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tio to the area APNF, it follows that it has no limit. Therefore there is likewiſe no limit which the area APrg may not exceed. When the figure APNF has an aſymptote BV parallel to the ordinates, the figure APrg has the ſame aſymptote: and it appears in the ſame manner, that the areas bounded by a given ordinate Ag, (which is ſuppoſed to be taken ſo near to B that the figure APrg has no aſymptote parallel to the ordinates betwixt A and B,) the aſymptote BV and the curves, either have both limits, or may be both produced till they exceed any given ſpace. Therefore what was ſhewn of the area APNF is applicable to the area APrg.

326. Cor. I. It is obvious that this is agreeable to what is ſhewn by the celebrated author of the Geometry of infinites, after his manner, in art. 1430. That when the baſe is infinitely produced, and the ordinate becomes an infiniteſimal of any order beneath the firſt, the area is finite; but when the ordinate is an infiniteſimal of the firſt order, whether it be a compleat infiniteſimal (as he expreſſes it) of that order or not, the area is infinite. For when PN is reciprocally as any power of BP if the exponent of this power exceed unit, PN becomes an infiniteſimal of an order beneath the firſt, according to this doctrine, when BP is ſuppoſed infinite; but if that exponent be unit or leſs than unit, PN is an infiniteſimal of the firſt order.

327. Cor. II. To give an example of this propoſition, let Pr be expreſſed by any fraction whoſe numerator and denominator conſiſt of terms formed from the powers of BP and given quantities, (as by $\frac{Ax^m + Bx^{m-1} + \&c.}{ax^n + bx^{n-1} + \&c.}$ where x is ſuppoſed to repreſent BP, m and n are any given numbers, A, B, a, b, &c. invariable quantities;) then, if the exponent of the higheſt power of BP (or x) in the denominator exceed the exponent of its higheſt power in the numerator, the baſe ſhall be an aſymptote of the figure APrg; and if this exceſs be greater than unit, (that is, if n exceed $m+1$,) there ſhall be a limit which the area bounded by the curve and aſymptote never can amount to; or, to make uſe of the uſual expreſſion, it ſhall have a finite value even when it is ſuppoſed to be produced infinitely: But if

if that exceſs be equal to unit, or leſs than unit, (that is, if n be equal to $m+1$, or leſs,) the area APrg ſhall have no ſuch limit; or, according to the uſual expreſſion, it ſhall become infinite when produced infinitely. If the loweſt exponent of BP (or x) in the terms of the denominator of that fraction exceed its loweſt exponent in the numerator, the figure ſhall have an aſymptote BV parallel to the ordinates; and when this exceſs is leſs than unit, there is a limit which the figure bounded by the curve and the aſymptote BV never amounts to; but when this exceſs is equal to unit, or greater than it, there is no ſuch limit: If any power of Pr be expreſſed by ſuch a fraction, then the exceſs we have deſcribed is to be divided by the exponent of this power, and the quotient in place of the exceſs is to be compared with unit. For let PN be reciprocally as the power of BP whoſe exponent is equal to this quotient, and the ratio of Pr to PN ſhall approach continually to an aſſignable limit when the figure is produced: becauſe a term in the value of Pr where BP is of any dimenſions, is to a term where it is of any lower dimenſions, as ſome power of BP is to a given quantity; and the latter of theſe terms is to be neglected in reſpect of the former, when the limit of the ratio of Pr to PN is required while the figure is produced along the baſe; but the former in reſpect of the latter, while it is produced along the aſymptote BV. And therefore the limit of the ratio of Pr to PN depends on thoſe terms only of the numerator and denominator of the value of Pr, where the exponents of BP are higheſt, in the former, and where theſe exponents are loweſt, in the latter caſe.

328. When the ratio of the fluxion of Pr to the fluxion of PN approaches continually to an aſſignable ratio, as its limit, while the figures APrg, APNF are produced along the aſymptote AP, the ratio of the ordinates Pr and PN to each other approaches likewiſe to an aſſignable ratio. For let Pſ and Pn the ordinates of the figures APſd, APnf meaſure the fluxions of Pr and PN reſpectively, the fluxion of the baſe being meaſured by the given line DG. Let AP be produced to any point q, and, Q being any where upon Pq, let the ordinates at Q and q meet the curves FN, gr, fn and dſ in M, V, T, Z, m, u, t and z; and let Pſ be at ſo great a diſtance from A that the figures FIG. 138.

AP*nſ*, AP*ſd* may have no aſymptote parallel to the ordinates beyond P*ſ*. The rectangles contained by P*r* and DG, and by PN and DG, are the limits to which the areas P*qzſ*, P*qtn* continually approach while P*q* is produced by art. 296. and the ratio of P*r* to PN is always the ratio of thoſe limits. The ratio of P*qzſ* to P*qtn* is always leſs than the greateſt ratio of the ordinate QZ to QT, and is greater than the leaſt ratio of QZ to QT, when this ratio is variable; and it coincides with this ratio when it is invariable: Therefore the ratio of P*qzſ* to P*qtn* always approaches to an aſſignable ratio as its limit while P*q* is produced, however great the diſtance AP may be; and, conſequently, the ratio of P*r* to PN approaches likewiſe to ſuch a limit. This is ſufficient for our purpoſe; but it might be ſhewn, that this limit can be no other than the limit of the ratio of P*ſ* to P*n*, or of the fluxion of P*r* to the fluxion of PN.

PROP. XXVIII.

329. *The fluxion of the baſe* AP *being given, let the fluxion of the ordinate* PR *be repreſented by* P*r*, *and the limit of the ratio of* P*r to an ordinate of an hyperbola that is reciprocally as a power of* BP *whoſe exponent is n, be aſſignable: Then the figure* APR*a ſhall have an aſymptote that is parallel to the baſe, or coincides with it, when n is greater than unit; and there ſhall be a limit which the area bounded by the curve a*R *and this aſymptote never can amount to, if n be greater than* 2.

FIG. 138. & 139.

Let A*a* parallel to the ordinates meet the curve in *a*, and *ap* parallel to the baſe meet PR in *p*. The figure *ap*R has an aſymptote parallel to the baſe when the area AP*rg* has a limit, by art. 298. that is, by the laſt propoſition, when the ratio of P*r* (which meaſures the fluxion of PR) to PN approaches to an aſſignable ratio as its limit, and PN is reciprocally as a power of BP whoſe exponent *n* exceeds unit. This aſymptote is determined by taking *a*K upon A*a* from *a* parallel to *p*R and in the

the ſame direction, ſo that the rectangle contained by *a*K and DG may be equal to the limit of the area AP*rg*, and, drawing KI parallel to the baſe, by art. 298. The aſymptote ſometimes coincides with the baſe, as when *p*R is taken from *p* towards P and *a*K is equal to A*a*. If the exponent *n* be not greater than unit, the figure has no aſymptote parallel to the baſe, by art. 298. & 325. In the ſame manner it appears, that the figure has an aſymptote parallel to the ordinates when *n* is equal to unit or greater; becauſe, AP being taken towards B, the figure AP*rg* may be produced in ſuch caſes till it exceed any given ſpace, by art. 325.

330. To demonſtrate the latter part of the propoſition, let KI be the aſymptote of the figure, that is parallel to the baſe when *n* exceeds unit, (by the laſt article;) let KI be taken upon CK produced towards O, and let P*n* meaſure the fluxion of PN, which is ſuppoſed to be reciprocally as a power of BP whoſe exponent exceeds unit. Then, (by art. 167.) the fluxion of the baſe being given, the fluxion of PN is as PN directly and BP inverſely; and, conſequently, P*n* is reciprocally as a power of BP whoſe exponent exceeds 2. Therefore, if the ratio of P*r* (that meaſures the fluxion of PR, or of IR) to a quantity that is reciprocally as the ſame power of BP approach to an aſſignable ratio, as its limit, the ratio of P*r* to P*n*, or of the fluxion of IR to the fluxion of PN, and (by art. 328.) the ratio of IR to PN, ſhall approach to ſuch a limit: And, becauſe there is a limit which always exceeds the area APNF, it follows, (by the 27th propoſition,) that there is likewiſe a limit which always exceeds the area *a*RIK. But if the exponent *n* be not greater than 2, the area *a*RIK may be produced till it exceed any given ſpace, by art. 325. becauſe the ratio of IR to a quantity that is reciprocally as a power of BP whoſe exponent is not greater than unit ſhall approach to an aſſignable ratio, as its limit, in this caſe. It appears in the ſame manner, that if KI be taken towards C and AP towards B, the area *a*RIK has a limit when *n* is leſs than 2, but has no limit when *n* is equal to 2, or greater. To give an example of this propoſition, if the ratio of the fluxion of the ordinate to the fluxion of the baſe be expreſſed by ſuch a fraction as was deſcribed

in art. 327. then, if the exceſs of the higheſt exponent of BP (or x) in the denominator above its higheſt exponent in the numerator be greater than unit, the figure has an aſymptote that is parallel to the baſe, or coincides with it: and if this exceſs exceed 2, there is a limit which the area bounded by the curve and this aſymptote can never amount to; but if the exceſs be equal to 2, or leſs than 2, that area may be produced till it exceed any given ſpace.

331. Cor. I. The firſt part of this propoſition, reduced to the ſtile of the method of infiniteſimals, agrees with the rule that was cited in art. 299. That when the figure is ſuppoſed to be produced infinitely, and the element of the ordinate becomes an infiniteſimal two degrees beneath the element of the baſe, the figure has an aſymptote that is either parallel to the baſe, or coincides with it. For, when the figure is ſuppoſed to be produced infinitely, the ratio of Pr to PN muſt be ſuppoſed to coincide with its limit; and, ſince PN is reciprocally as a power of BP whoſe exponent is greater than unit, and the fluxion of the ordinate PR is to the fluxion of the baſe as Pr is to DG, if theſe fluxions be repreſented by infinitely ſmall elements of the ordinate and baſe, the element of the ordinate muſt be to the element of the baſe as a given quantity is to a quantity that is expreſſed by a power of the infinite line BP whoſe exponent exceeds unit. This rule is chiefly of uſe (as the celebrated author obſerves) when the curve is not geometrical, and when AP and PR do not both enter the quantities that expreſs the relation of their fluxions; for if PR, or any other variable quantity beſides the baſe AP and its powers, enter the value of the fluxion (or of the element) of PR, regard muſt be had to the ratio of that variable quantity to the baſe AP when it is ſuppoſed infinite: And in applying this rule, the various orders of infinites, and infiniteſimals, are beſt determined by the powers of the baſe, and their reciprocals. The continuation of this rule, and of that given in art. 326. will appear from what follows.

332. Cor. II. The ſecond part of this propoſition may be expreſſed, according to the method of infiniteſimals, thus: The figure apR has an aſymptote parallel to the baſe, and the area bounded by the curve and that aſymptote is finite, though infi-

nitely

nitely produced, when, the baſe being ſuppoſed infinite, the element of the ordinate *p*R becomes an infiniteſimal of an order three or more degrees beneath the element of the baſe, which is always ſuppoſed an infiniteſimal of the firſt order; but that area is infinite, when the element of the ordinate is an infiniteſimal of an order that is only two degrees beneath that of the element of the baſe.

333. The fluxion of the baſe being given, let the ratio of the quantity that meaſures the ſecond fluxion of the ordinate PZ, to a quantity that is reciprocally as a power of BP whoſe exponent is *n*, approach to any aſſignable ratio as its limit. Then the figure has an aſymptote that is either oblique to the baſe, or parallel to it, or that coincides with it, when *n* exceeds 2; and there is a limit which the area bounded by the curve and this aſymptote never amounts to, when *n* exceeds 3. This is demonſtrated from art. 300. 328. & 330. in the ſame manner as the laſt propoſition was demonſtrated from art. 298. 325 & 328. The aſymptote is oblique to the baſe when the ratio of the firſt fluxion of the ordinate to the fluxion of the baſe approaches to an aſſignable ratio, as its limit; but it is parallel to the baſe, or coincides with it, when this limit is not aſſignable. If *n* be equal to 2, or leſs, the figure has not ſuch an aſymptote; and if *n* be equal to 3, or any number betwixt 2 and 3, the area bounded by the curve and aſymptote may exceed any limit. FIG. 140.

334. In order to comprehend the continuation of thoſe theorems in one view, let us call the aſymptote, of the *firſt order*, when it coincides with the baſe of the figure; of the *ſecond order*, when it is a right line parallel to the baſe; of the *third order*, when it is a right line oblique to the baſe; of the *fourth order*, when it is a common parabola that has its axis perpendicular to the baſe; and, in general, of the order $r+2$, when it is a parabola the ordinate of which is always as a power of the baſe whoſe exponent is *r*. Then let PZ be the ordinate of the figure, and let its fluxion of any order expreſſed by *m* (that is, its firſt fluxion when *m* is unit, its ſecond fluxion when *m* is 2, and ſo on) be repreſented by P*r*, the fluxion of the baſe being given. Let PN be reciprocally as any power of BP whoſe exponent is any number *n*; and let the ratio of P*r* to PN ap-

proach

proach continually to any aſſignable ratio, as its limit, while BP is produced. Then, if the number n be greater than m, the figure ſhall have an aſymptote of the order expreſſed by $m+1$, or of an order that is expreſſed by a leſſer number. When the exceſs of n above m is greater than unit, there is a limit which the area bounded by the curve aZ and that aſymptote never can amount to, while BP is produced; when that exceſs is equal to unit, or leſs, this area may be produced till it exceed any given ſpace: And if n be equal to m, the figure has no ſuch aſymptote. Let the figure be now produced on the other ſide of AF, or AP be taken from A towards B, and when n is equal to m, or greater, the figure has an aſymptote parallel to the ordinates, that paſſes through B; when n is equal to m, or when the exceſs of n above m is leſs than unit, there is a limit which the area bounded by the curve and this aſymptote never can amount to; when that exceſs is equal to unit, or greater, this area may be produced till it exceed any given ſpace: But when n is leſs than m, the figure has not an aſymptote through B parallel to the ordinates.

335. The firſt part of the laſt article, reduced to the ſtile of the method of infiniteſimals, ſhews, that when BP is ſuppoſed to be produced infinitely, and the *difference* of the ordinate of the order expreſſed by m becomes an infiniteſimal of an order that is as many degrees beneath that of the element (or firſt *difference*) of the baſe as there are units in $2m$, then the curve has an aſymptote of the order expreſſed by $m+1$, or of ſome inferior order; when that *difference* becomes an infiniteſimal of an order that is as many degrees beneath that of the element of the baſe as there are units in $2m+1$, the area bounded by the curve and aſymptote is finite though it be produced infinitely; but when the order of that *difference* is only $2m$ degrees beneath that of the element of the baſe, then the area is infinite: And this ſhews the continuation of the rules given by the learned author above mentioned. Thus, for example, the baſe being produced infinitely, when the ſecond *difference* of the ordinate is of an order four or more degrees beneath that of the element of the baſe, the figure has a rectilineal aſymptote that is oblique to the baſe, or is parallel to it, or coincides with it; but if this

dif-

difference be of an order that is not ſo many degrees beneath that of the element of the baſe, the curve has not ſuch an aſymptote. If the ſecond *difference* of the ordinate is of an order five or more degrees beneath that of the element of the baſe, the area bounded by the curve and aſymptote is finite though produced infinitely; but if it is not of an order that is five degrees beneath that of the element of the baſe, the area is infinite.

336. It is eaſy to ſee from art. 307. & 309. when the ſolids generated by hyperbolic areas revolving about their aſymptotes have limits, and when they may be produced till they exceed any given ſolid. If *a*ME be any hyperbola, the ſolid generated by APNF has a limit, or not, according as AP is taken upon BA produced beyond A, or from A towards B, by art. 301. and ſince IM the ordinate of the hyperbola *a*ME to its aſymptote KI is reciprocally as ſome power of BP, its fluxion is reciprocally as a power of BP whoſe exponent exceeds unit, and, the ſquare of PN being always as this fluxion, (art. 307.) PN muſt be as ſome power of BP whoſe exponent is greater than $\frac{1}{2}$. Therefore, when PN is reciprocally as a power of BP whoſe exponent is greater than $\frac{1}{2}$, the ſolid generated by APNF about AP has a limit, or not, according as AP is taken upon BA produced beyond A, or from A towards B. When *a*ME is a logarithmic that has BV for its aſymptote, the fluxion of PM is reciprocally as BP, and PN is reciprocally as the power of BP whoſe exponent is $\frac{1}{2}$; and the ſolid generated by APNF about AP has no limit in either caſe, becauſe the cylinder generated by the rectangle *ag* may exceed any given ſolid. When *a*ME is a parabola that touches BV in L, the fluxion of PM is reciprocally as a power of BP whoſe exponent is leſs than unit, and PN is reciprocally as a power of BP whoſe exponent is leſs than $\frac{1}{2}$: And in this caſe, when AP is taken from A towards B, the ſolid generated by APNF has a limit which it can never amount to, *viz.* the cylinder generated by the rectangle *a*R about the axis *ak*; but it may exceed any given ſolid when AP is taken on the other ſide of A. When PN is reciprocally as a power of BP whoſe exponent is any fraction greater than $\frac{1}{2}$, but not greater than unit, and AP is taken upon

Fig. 114. Fig. 114. n. 2. Fig. 115. Fig. 114.

on BA produced from A, there is no limit which the area APNF may not exceed: but there is a limit which the ſolid generated by this area never can amount to; and therefore, in this caſe, when the figure is ſuppoſed to be produced infinitely, an infinite area is ſaid to generate a finite ſolid. If that exponent be ſtill greater than $\frac{1}{2}$, but leſs than unit, and AP be taken towards B, there is a limit which always exceeds the area APNF, but there is none which always exceeds the ſolid generated by it; and, in this caſe, when the figure is ſuppoſed to be produced infinitely, a finite area is ſaid to generate an infinite ſolid. In the former caſe, the fluxions of the ſolid and area both decreaſe while AP is produced; but the fluxion of the ſolid decreaſes faſter, and is meaſured by a figure which is to the ſolid contained by the rectangle P*b*, that meaſures the fluxion of the area, and by the given line AF, in a ratio that, by producing AP, becomes leſs than any given ratio; and hence it is not ſurpriſing, that the ſolid ſhould have a limit, though the area has none. In the latter caſe, that ratio by producing the figures becomes greater than any given ratio; and it is therefore eaſy to conceive, that though the area has a limit, the ſolid may have none. What has been ſhewn of the ſolid generated by the area APNF, is to be extended to the ſolid generated by the area AP*rg* when the ratio of P*r* to PN (however variable it may be) approaches continually to an aſſignable ratio, as its limit, while the figures are produced along their aſymptotes. And, in general, what was ſhewn of areas, may be transferred, with ſome neceſſary precautions, to ſolids. For example, the fluxion of the baſe being given, if the ratio of the fluxion of PR to a quantity that is reciprocally as a power of BP whoſe exponent is n, approach to an aſſignable ratio, as its limit, and n exceed 2, there is a limit which always exceeds the ſolid generated by the area *a*RIK about the axis AP, which is ſuppoſed to be taken upon BA produced from A.

FIG. 138.

337. There are ſeveral other methods by which it may be diſcovered when a figure has an aſymptote, and of what kind it is. When the value of the ordinate is reſolved into a ſeries that converges the faſter the greater the baſe is, the aſymptote may be determined from the firſt terms of the ſeries when they are

ſuch

ſuch as remain invariable, or continually increaſe, while the baſe is produced. If the difference betwixt the ſubtangent and the abſciſs approach continually to a finite right line, as its limit, while the branch of the curve is produced, and the baſe (or the ordinate) may increaſe till it exceed any given right line, it is obvious that the figure muſt have a right line for its aſymptote, that meets the baſe at a diſtance from the beginning of the abſciſs equal to that limit, by art. 313. When the curve has a common parabola for its aſymptote, the ratio of the ſubtangent to the abſciſs approaches continually to that of 2 to 1, when the axis of the parabola coincides with the baſe; but to that of 1 to 2, when the axis is perpendicular to the baſe: and by obſerving the limit to which the ratio of the ſubtangent and abſciſs approaches, parabolic aſymptotes, of various kinds, may be diſcovered. When the logarithmic NF is continued on the other ſide of AF, this ratio decreaſes and may become leſs than any given ratio, and no parabolic figure can be its aſymptote. FIG. 116.

338. If SP perpendicular from a given point S upon LP the tangent of the curve LB approach continually in poſition and magnitude to any aſſignable right line, and SL may increaſe without end, it is obvious that the curve BL muſt have a rectilineal aſymptote. If ST, which is perpendicular on PT the tangent of the curve DP, (and is a third proportional to SL and SP,) approach in the ſame manner to a finite right line, the curve DP has a right line, and BL a common parabola, for its aſymptote. If SK perpendicular on TK approach thus to an aſſignable limit, the curve GT has a right line, DP a common parabola, and BL a ſemicubical parabola, for their aſymptotes. In general, when SL may be increaſed without end, if the ratio of SL to a right line that is as any power of SP whoſe exponent is greater than unit, approach to an aſſignable ratio, as its limit, a parabola, the ordinate of which is always as the ſame power of the abſciſs, may be the aſymptote of the curve BL; but if the ratio of SL to SP approach to ſuch a limit, the figure has no rectilineal or parabolic aſymptote. In the ſame manner, if the ratio of the fluxion of the angle ASP to the fluxion of the angle ASL approach to the ratio of any number m to a greater number n, while BL is produced, a parabola whoſe ordinate is al- FIG. 103.

ways as the power of the abciſs of the exponent $\frac{m}{n}$ is the aſymptote of the curve BL. There are ſtill other rules by which it may be ſometimes diſcovered when a figure has an aſymptote and the area a limit, but we have inſiſted on this ſubject already at a ſufficient length.

PROP. XXIX.

339. *The figure* APNF *being ſuppoſed to revolve about its aſymptote* AP, *there is a limit which always exceeds the ſurface generated by the curve* FN, *when there is a limit which always exceeds the area* APNF; *but when this area may be produced till it exceed any given ſpace, the ſurface generated by* FN *has no limit.*

Fig. 141.

Let N*t* be the tangent at N, and let *xt* parallel to PN meet N*t* in *t*, and meet N*x* parallel to the baſe and equal to DG (which meaſures the fluxion of the baſe) in *x*. The fluxion of the ſurface generated by FN is meaſured by a ſpace that is to the rectangle contained by PN and N*t* in the invariable ratio of the circumference of a circle to its radius; this rectangle is always greater than the rectangle P*x* which meaſures the fluxion of the area APNF, becauſe N*t* is always greater than N*x*: Therefore the ſurface generated by FN is always greater than the area APNF; and, conſequently, if this area may be produced till it exceed any given ſpace, the ſurface FN may likewiſe be produced till it exceed any given ſpace: which is one part of the propoſition. Let FT the tangent at F meet the aſymptote in T, and Ag be to AF as FT is to AT; let P*r* be to PN in the invariable ratio of Ag to AF, and *rz* parallel to the baſe meet *tx* produced in *z*. Becauſe the curve is convex towards its aſymptote, *tx* continually decreaſes while the figure is produced, (art. 184.) and the ratio of N*t* to N*x* decreaſes, and is always leſs than the ratio of FT to AT or of P*r* to PN; and the rectangle contained by PN and N*t* is leſs than the rectangle

Plate V. Pa. 282.

Fig. 132.

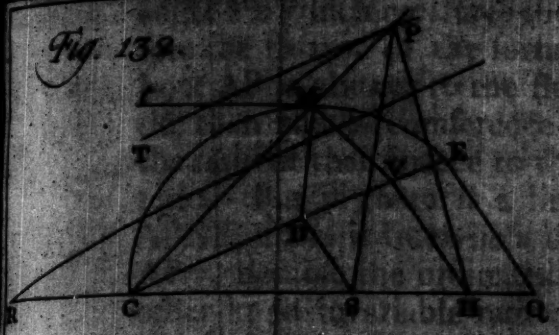


Fig. 133.

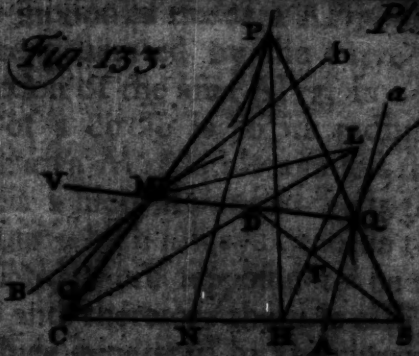


Fig. 134.

Fig. 135.

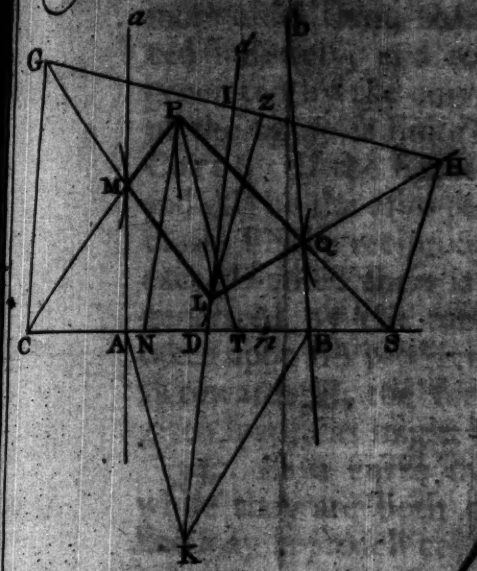


Fig. 137.

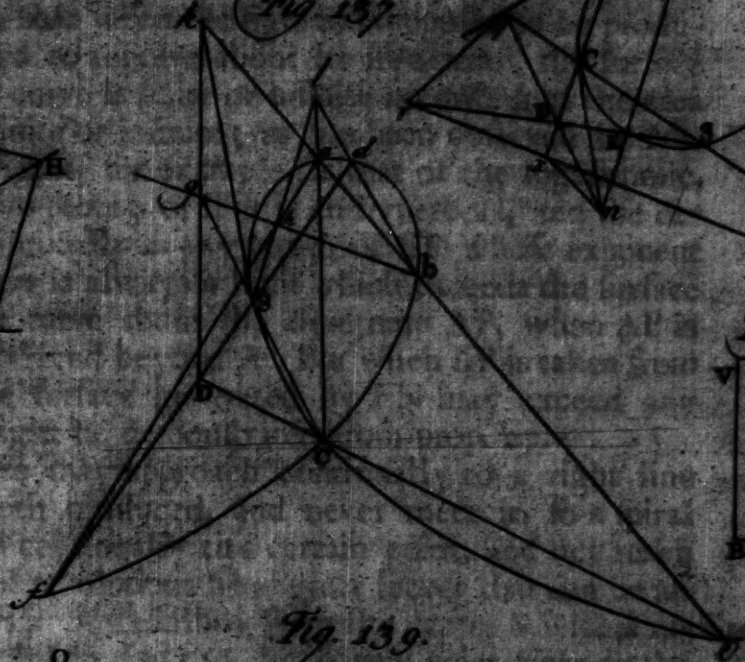


Fig. 136.

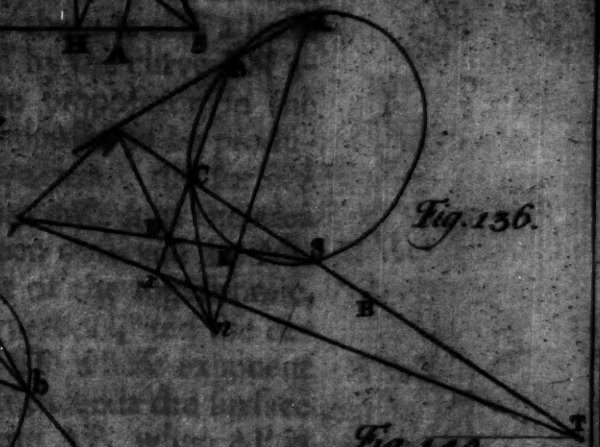


Fig. 140.

Fig. 138.

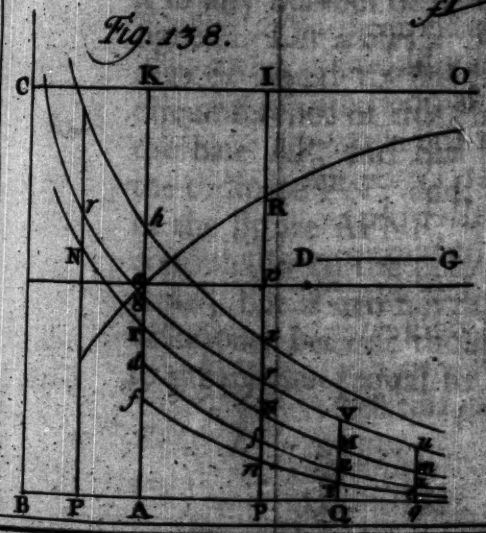


Fig. 139.

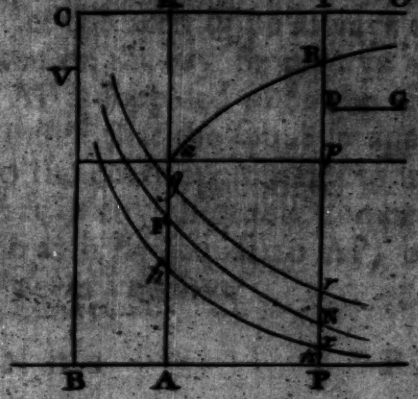
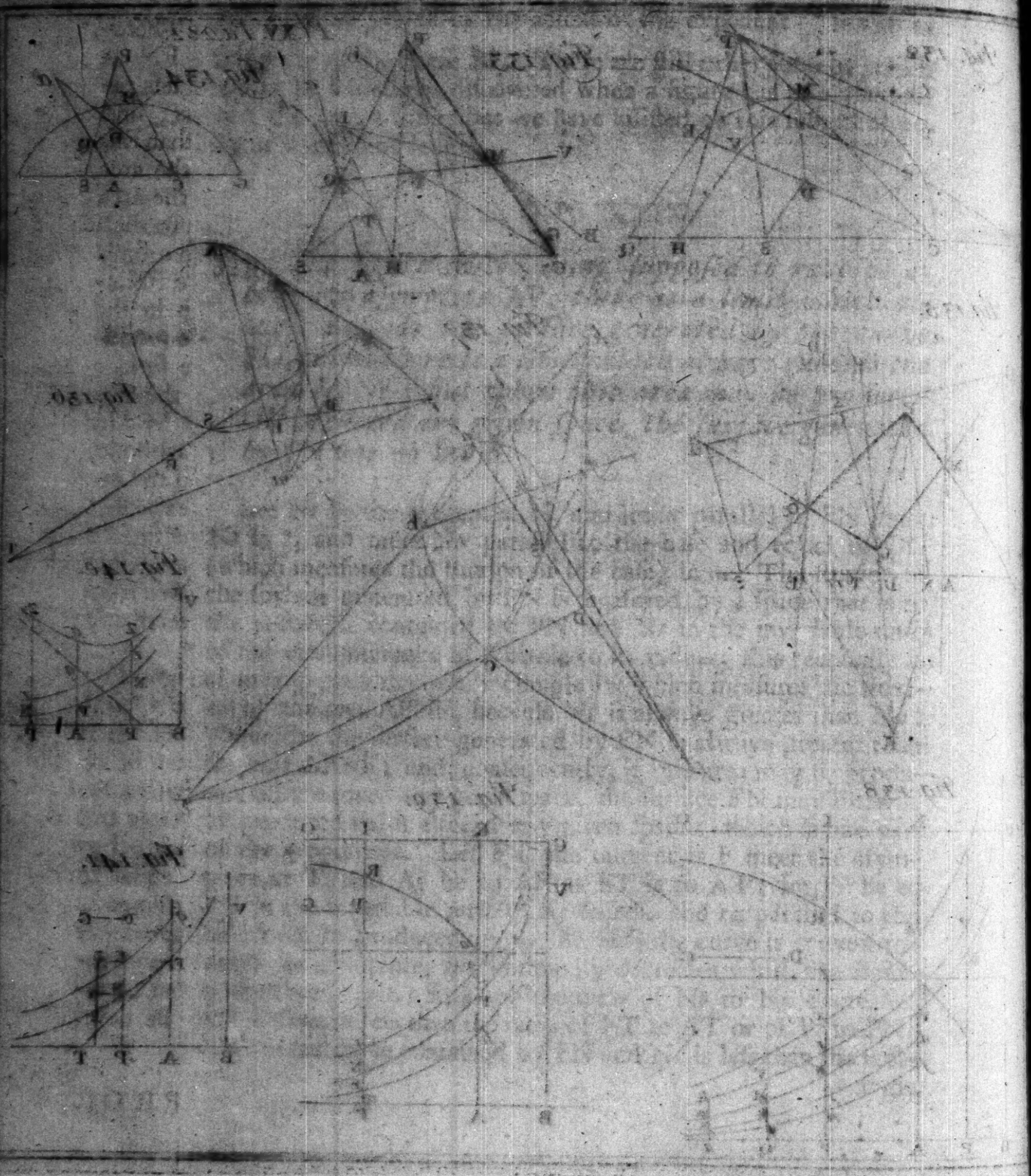


Fig. 141.



angle P*z* which meaſures the fluxion of the area AP*rg*. Therefore the fluxion of the ſurface generated by the curve FN about AP is always to the fluxion of the area AP*rg*, in a leſs ratio than the circumference of a circle is to its radius; and the ratio of that ſurface to the area AP*rg* is always leſs than this ratio. But if there is a limit which always exceeds the area APNF, there is likewiſe a limit which always exceeds the area AP*rg*, becauſe the ordinates PN, P*r* are ſuppoſed to be to each other in an invariable ratio; and therefore there is alſo a limit which always exceeds the ſurface generated by the curve FN about AP. Thus it appears, to expreſs the propoſition in the more uſual form, that when the figure is ſuppoſed to be produced infinitely, and to revolve about its aſymptote, the ſurface generated by the curve is finite or infinite according as the area of the figure is finite or infinite; as has been obſerved by Mr. Cotes. If the figure, for example, is that of the logarithmic, and AP be the aſymptote; or if it is an hyperbola, and the ordinate PN be reciprocally as any power of BP whoſe exponent exceeds unit: there is always a limit which exceeds the ſurface generated by the curve about the aſymptote AP, when AP is taken upon BA produced beyond A: But when AP is taken from A towards B, the ſurface generated by FN may exceed any given ſpace, as might be demonſtrated from prop. 27.

340. As a curve may approach continually to a right line while they are both produced, and never meet it, ſo a ſpiral line may approach continually to a certain point, and not reach it in any number of revolutions how great ſoever that can be aſſigned. Let a circle be deſcribed from the center S with a radius S*b* equal to AF in fig. 114. let *b* be a point given in the circumference of this circle; let the arch *br* be always equal to the baſe AP, and S*m* be taken on the ray S*r* always equal to the ordinate PN; and, ſince AP is ſuppoſed to be the aſymptote of the figure APNF, it is manifeſt that the ſpiral *br* approaches continually to the point S, as the curve approaches to its aſymptote; but it can never reach it in any aſſignable number of revolutions, becauſe though the arch *br* (equal to AP) be never ſo great, S*m* (equal to PN) never vaniſhes. Fig. 142. & 114.

PROP. XXX.

341. *The conſtruction in art.* 307. *being ſuppoſed, let the arch hr be always equal to* AP, *and* S*m equal to* PN; *and the ſpiral area* S*hm ſhall be always to the rectangle ag in the invariable ratio of* DG *to* H*h the diameter of the circle hr*H.

FIG. 142. & 114.

By what was ſuppoſed in art. 307. the ſquare of PN is to the ſquare of DG as the fluxion of PM is to the fluxion of AP, or (if M*l* parallel to the baſe be equal to DG, and *lt* parallel to the ordinates meet the tangent M*t* in *t*) as *lt* is to M*l*, or DG; and therefore the ſquare of PN is equal to the rectangle contained by DG and *lt*. The fluxion of the ſpiral area S*hm* is to the fluxion of the ſector S*hr* (by prop. 5.) as the ſquare of S*m* is to the ſquare of S*h*. The fluxion of the baſe AP, or of the arch *hr*, being meaſured by DG, the fluxion of the ſector S*hr* is meaſured by one half of the rectangle contained by S*h* and DG, and is to the fluxion of the rectangle *ag* (which is meaſured by the rectangle contained by *au*, or DG, and *lt*) as S*h* is to 2*lt*. Therefore the fluxion of the ſpiral area S*hm* is to the fluxion of the rectangle *ag* as the ſquare of S*m*, (or of PN,), or the rectangle contained by DG and *lt*, is to the rectangle contained by S*h* and 2*lt*; that is, as DG is to 2S*h* or 2AF: and, this ratio being invariable, it follows, that the ſpiral area S*hm* is to the rectangle *ag* as DG is to H*h* the diameter of the circle *hr*H. This propoſition obtains, whether AP and KI be aſymptotes of the curves FN, *a*M or not; it is ſufficient that the ſquare of PN be to the ſquare of DG as the fluxion of PM is to the fluxion of AP, that *hr* be always equal to AP, S*m* equal to PN, and that PM either increaſe or decreaſe continually.

342. COR. I. Becauſe the ſector S*hr* is equal to one half of the rectangle FP, it follows, that the ſpiral area S*hm* is to the ſector S*hr* as the ſolid contained by *ag* and DG to the ſolid contained by FP and S*h*, or AF, and therefore as the cylinder generated by *ag* about *ax*, or the ſolid generated by APNF about AP,

AP, to the cylinder generated by FP about AP. Thus, when FN is a right line, *hr* is the ſpiral of Archimedes, the area S*hm* is to the ſector S*hq* as the fruſtum of a cone generated by the trapezium APNF about the axis AP is to the cylinder generated by the rectangle FP about the ſame axis; and if S*y* be the tangent of the ſpiral at S, the ſpiral area S*wmh* ſhall be to the ſector S*hy* as one is to three; as was ſhewn after ARCHIMEDES in the introduction. FIG. 142. n. 2.

343. COR. II. When there is a limit which always exceeds the ſolid generated by the area APNF about AP, there is likewiſe a limit which always exceeds the ſpiral area S*hm* generated by the ray S*m* while it revolves about S; and this limit is to the rectangle *ad* as DG is to H*b*. But if the ſolid generated by the area APNF produced may exceed any given ſolid, the area that is generated by the ray S*m*, while it revolves continually about S, may likewiſe exceed any given ſpace. In the ſame manner, S*h* being equal to *a*K, if *kr* be always equal to *ap*, and S*m* to *p*M, the area generated in this caſe by *rm* ſhall have a limit, or not, according as the ſolid generated by the area *a*MIK about the axis *ap* has a limit, or may be produced till it exceed any given ſolid: And in this caſe the circle *hr*H is the aſymptote of the ſpiral. If S*m* be always equal to PZ in fig. 140. the ſpiral of Archimedes is the aſymptote of the ſpiral *hm*: and this might be carried further by art. 334. *&c.*

344. COR. III. Let FN*e* be a common hyperbola, BA and BC its aſymptotes, and *hm* ſhall be the curve that is called the *reciprocal* or *hyperbolic ſpiral*; let the arch *hb* be equal to BA, join S*b*, let a circle deſcribed from the center S through *m* meet S*b* in *o*, and the arch *mo* ſhall be of an invariable magnitude equal to *hb*, or BA; for *br* is to *mo* as S*b* is to S*m*, or AF to PN, and therefore as BP (or *br*) to BA. The ſpiral area S*mb* is to the rectangle *ag* as DG is to 2AF, and therefore in this caſe S*mb* is equal to one half of the rectangle contained by BA and the difference of AF and PN, or by *hb* and *mr*; and the limit of this area is equal to the ſector S*hb*. Let ST perpendicular to S*m* meet the tangent of the ſpiral in T, and ST ſhall be of an invariable magnitude equal to *hb*, or BA; becauſe S*m* is to ST in the ratio compounded of that of the fluxion of S*m* FIG. 143. & 114.

to

to the fluxion of *br* (or AP) and of the ratio of S*b* (or AF) to S*m*; that is, in the ratio of AF to BP, or of PN (or S*m*) to BA. It is obvious, that a right line *al* parallel to S*b*, at a diſtance S*a* from it equal to BA on the ſame ſide that *b* is from *b*, is an aſymptote of this ſpiral continued without the circle *bb*.

345. Cor. IV. Let Sp perpendicular from S on *m*T the tangent of the hyperbolic ſpiral (deſcribed in the laſt corollary) meet it in p, and the point p ſhall be always found in the ſpiral whoſe rays decreaſe *proportionally* while the length of the curve *ap* increaſes uniformly, which is called by Mr. Cotes the *complicated tractrix*. For if the angle Spt be made equal to S*m*T, pt ſhall be the tangent of the curve in which p is always found, by art. 211. and if S*t* perpendicular to S*a* meet *pt* in *t*, the triangles p*St*, Sp*t* ſhall be ſimilar and equal, and *pt* the tangent of this ſpiral is always equal to the invariable line ST. The fluxion of this ſpiral *ap* is to the fluxion of the ray Sp as the invariable tangent *pt* (or ST) is to the ray Sp, and therefore Sp decreaſes *proportionally* while *ap* increaſes uniformly. Hence the points of this ſpiral may be found by taking S*m*, upon any ray S*n*, in the ſame ratio to S*n* as the given arch *bb* is to the arch *br*, conſtituting ST perpendicular on S*m* equal to the invariable line ST, or to the arch *bb*, joining *m*T and drawing Sp perpendicular to *m*T in p. Or, let a circle *kun* be deſcribed with the radius *ſk* equal to ST, and let *ſz* meet the tangent *kz* in *z* and the circle in *u*; let *ux* be perpendicular on S*k* in *x*; and, if the arch *ky* be equal to the exceſs of the tangent *kz* above the arch *ku*, and S*n* be taken upon S*y* always equal to S*x*, (the coſine of *ku*,) then *n* ſhall be a point in this ſpiral. A conſtruction of this ſpiral is given by Mr. Cotes, *Harmon menſur. p.* 84. It is likewiſe conſidered by Mr. Varignon, *Mem. de l'acad. roy.* 1704. All thoſe ſpirals in which the ratio of the fluxion of the curve to the fluxion of the ray from S, is the ſame as the ratio of ſome power of the given line S*b* to the ſame power of that ray, are conſtructed from this ſpiral by taking the angle *b*S*q* to *b*Sp as unit is to the exponent of that power, and S*q* ſo that the ratio of the ſame powers of S*q* and S*b* may be that of Sp to S*b*; for *q* ſhall be found in ſuch a ſpiral.

Fig. 144.

Fig. 143.

PROP.

PROP. XXXI.

346. *When there is a limit which always exceeds the area* APNF *while it is produced along the aſymptote* AD, *there is likewiſe a limit which always exceeds the length of the ſpiral line* hm *while it approaches to* S; *but if the area* APNF *produced may exceed any given ſpace, the ſpiral line* hm *may be continued till it exceed any given line.* FIG. 145.

Let *m*T touch the ſpiral in *m*, and meet ST perpendicular to S*m* in T. The fluxion of the baſe AP, or of the arch *br*, being repreſented by any given right line DG, let P*n* the ordinate of the figure AP*nf* meaſure the fluxion of PN or S*m*; let P*r*, the ordinate of the figure AP*rg*, be always to PN in the invariable ratio of DG to SH; let the ſquare of P*q*, the ordinate of the figure AP*qd*, be always equal to the ſum of the ſquares of P*r* and P*n*; and, *qx* parallel to the baſe being made equal to DG, complete the rectangle P*x*. Let a circle deſcribed from the center S through *m* meet S*b* in *o*; and, while *m* proceeds in the ſpiral, the fluxion of *om* ſhall be to the fluxion of *br* (or of AP,) which is meaſured by DG, as S*m* (or PN) is to S*b*; but P*r* is to DG as PN is to S*b*: Therefore the fluxion of *om* is repreſented by P*r*; and, ſince it is to the fluxion of S*m* as ST is to S*m* (by prop. 16.) it follows, that ST is to S*m* as P*r* is to P*n*, and *m*T to S*m* as P*q* is to P*n*. Therefore the fluxion of the ſpiral line *hm* is to the fluxion of S*m* as P*q* is to P*n*, and to the fluxion of AP as P*q* is to DG; and the rectangle contained by DG and the right line that meaſures the fluxion of *hm* is always equal to the rectangle P*x* that meaſures the fluxion of the area AP*qd*. Therefore the rectangle contained by the ſpiral line *hm* and the given right line DG is always equal to the area AP*qd*. But P*q* is always leſs than the ſum of P*n* and P*r*, becauſe the ſquare of P*q* is leſs than the ſquare of the ſum of P*n* and P*r*, by the ſuppoſition; and, conſequently, the area AP*qd* is always leſs than the ſum of the areas AP*rg*, AP*nf*. The rectangle contained

tained by AF and DG is always greater than the area AP*nf*, by art. 296. Therefore the rectangle contained by the ſpiral line *hm* and DG is always leſs than the ſum of the area AP*rg* and the rectangle contained by AF, DG. Becauſe P*r* is to PN in the invariable ratio of DG to SH, the area AP*rg* is to the area APNF in the ſame ratio; and if there is a limit which always exceeds the area APNF, there muſt likewiſe be a limit which always exceeds the area AP*rg*; and there is conſequently a limit which exceeds the rectangle contained by the ſpiral *hm* and DG. Therefore there is a right line which always exceeds the ſpiral *hm*. If, by producing the figure, the area APNF may exceed any given ſpace, the area AP*rg*, and the rectangle contained by the ſpiral *hm* and DG, (which is always greater than AP*rg*,) may likewiſe exceed any ſpace, and the ſpiral *hm* may exceed any given line.

347. Cor. I. This theorem, expreſſed according to the more uſual form, is, That when the figure APNF is produced infinitely, and the ſpiral *hm*, after having made an infinite number of revolutions, reaches the point S, the length of the ſpiral is finite or infinite, according as the area APNF is finite or infinite. When it is expreſſed in this manner, it is one of the paradoxes of this kind that has the moſt myſterious appearance; but there is nothing more wonderful in it, in the manner it is here propoſed, than that a line may be continually increaſing, and the increments acquired by it decreaſe in ſuch a manner that it ſhall never amount to a given line. See art. 289.

348. Cor. II. If the ray S*m* be reciprocally as any power of the arch *br* whoſe exponent exceeds unit, or if the ratio of S*m* to a quantity that is reciprocally as ſuch a power of *br* approach to an aſſignable ratio, as its limit, while the figure is produced, there is a certain limit which always exceeds the length of the ſpiral *hm*. But if S*m* be reciprocally as a power of the arch *br* whoſe exponent is equal to unit, or leſs, the ſpiral *hm* may be continued till it exceed any given line.

349. Cor. III. When FNe is the logarithmic curve, S*m* decreaſes *proportionally* while *br* increaſes uniformly, and the fluxion of S*m* (or PN) is to the fluxion of *br* (or AP) as S*m* is to an invariable line S*c* that is equal to the ſubtangent of the logarithmic.

garithmic. The fluxion of *br* is to the fluxion of *om* as S*b* is to S*m*; and therefore the fluxion of S*m* is to the fluxion of *om* as S*b* is to S*r*; and, S*m* being to ST in the ſame ratio, the angle S*m*T is invariable. Therefore the fluxion of the ſpiral *bm* is to the fluxion of *rm* in the invariable ratio of *m*T to S*m*; and if *rt* perpendicular to S*r* meet T*m* produced in *t*, the ſpiral line *bm* ſhall be equal to the right line *mt*; and *m*T is the limit to which the ſpiral line produced from *m* continually approaches. This curve is called the *logarithmic* ſpiral.

350. As a right line, or figure, may increaſe continually and never amount to a given line, or area; ſo there are progreſſions of fractions which may be continued at pleaſure, and yet the ſum of the terms be always leſs than a certain finite number. If the difference betwixt their ſum and this number decreaſe in ſuch a manner, that by continuing the progreſſion it may become leſs than any fraction how ſmall ſoever that can be aſſigned, this number is the *limit of the ſum of the progreſſion*, and is what is underſtood by the value of the progreſſion when it is ſuppoſed to be continued infinitely. Theſe limits are analogous to the limits of figures which we have been conſidering, and they ſerve to illuſtrate each other mutually. The areas of figures cannot be expreſſed in many caſes but by ſuch progreſſions; and when the limits of figures are known, they may be ſometimes applied with advantage for approximating to the ſums of certain progreſſions. Let the terms of any progreſſion be repreſented by the perpendiculars AF, BE, CK, HL, *&c.* that ſtand upon the baſe AD at equal diſtances; and let PN be any ordinate of the curve FN*e* that paſſes through the extremities of thoſe perpendiculars. Suppoſe AP to be produced; and according as the area APNF has a limit which it never amounts to, or may be produced till it exceed any given ſpace, there is a limit which the ſum of the progreſſion never amounts to, or it may be continued till its ſum exceed any given number. For let the rectangles FB, EC, KH, LI, *&c.* be completed, and, the area APNF being continued over the ſame baſe, it is always leſs than the ſum of all thoſe rectangles, but greater than the ſum of all the rectangles after the firſt. Therefore the area APNF and the ſum of thoſe rectangles either both have limits, or both have none; and it is FIG. 146.

obvious, that the same is to be said of the sum of the ordinates AF, BE, CK, HL, &c. and of the sum of the terms of the progression that are represented by them. If the curve FN*e*, for example, be the common hyperbola, *b* the center, *b*P the asymptote; and, AB being equal to *b*A, if AF represent unit, the series of ordinates shall represent the progression 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. which therefore may be continued till it exceed any given number, as the hyperbolic area may be produced till it exceed any given space, by art. 297. But if FN*e* be an hyperbola of any higher order, so that the ordinate PN be reciprocally as any power of the base *b*P whose exponent is greater than unit, the area APNF, and the sum of the progression that is represented by the series of ordinates, shall have limits. Thus there is always a limit of the sum of the fractions that have unit for their common numerator, and the squares, cubes, or any other powers of thenumbers 1, 2, 3, 4, &c. whose exponents exceed unit, for their successive denominators.

351. And here we may apply what was shewn above, in art. 325. &c. concerning areas, to progressions. If the ratio of the last term of the progression to a fraction that is reciprocally as any power of the number represented by *b*P, approach to an assignable ratio, as its limit; then there is a limit of the sum of the progression, or not, (or, as it is usually expressed, the value of the progression continued infinitely is finite, or infinite,) according as the exponent of that power exceeds unit, or not. If the terms of the progression are such as arise by substituting successively any numbers that increase by equal differences in place of x in the quantity $\frac{Ax^m + Bx^{m-1} + \&c.}{ax^n + bx^{n-1} + \&c.}$ described in art. 327. the progression shall always have a limit when the excess of n above m is greater than unit, but it may be continued till the sum of the terms exceed any given number when this excess is equal to unit, or less. Thus, if the figurate numbers * of any order

* The figurate numbers of the first order are 1, 1, 1, 1, 1, &c. These of the second order are the successive sums of those of the first order, *viz.* 1, 2, 3, 4, 5, &c. and form an arithmetical progression. These of the third order are the

order be divided ſucceſſively by the correſponding figurate numbers of any order that is two or more degrees higher, the ſum of the fractions that ſhall ariſe in this manner ſhall have a limit; but if they are divided by the figurate numbers of the next ſuperior order, the progreſſion of the quotients may be continued till it exceed any given number. In like manner it follows from what was ſhewn in art. 329. that when the terms of a progreſſion increaſe, but their ſucceſſive differences decreaſe, and the ratio of the laſt difference to a number that is always reciprocally as any power of *b*P approaches to an aſſignable ratio, as its limit; then there is a limit which always exceeds the terms of this progreſſion, or not, according as the exponent of that power is greater than unit, or not. If this exponent be greater than 2, and the terms be ſubducted ſucceſſively from their limit, a new progreſſion ſhall be formed the ſum of which ſhall have a limit.

352. When the area APNF has a limit, we not only conclude from this, that the ſum of the progreſſion repreſented by the ordinates has a limit; but when the former limit is known,

the ſucceſſive ſums of thoſe of the ſecond, *viz.* 1, 3, 6, 10, 15, *&c.* and are the triangular numbers. Theſe of the fourth order are the ſucceſſive ſums of thoſe of the third, *viz.* 1, 4, 10, 20, 35, *&c.* and are the pyramidal numbers, and ſo on. The figurate numbers of any order may likewiſe be found, without computing thoſe of the preceeding orders, by taking the ſucceſſive products of as many of the numbers 1, 2, 3, 4, 5, *&c.* in their natural order, as there are units in the number which denominates the order of figurates required, and dividing always thoſe products by the firſt product. Thus the triangular numbers are found by dividing the products 1×2, 2×3, 3×4, 4×5, 5×6, *&c.* each by the firſt product 1×2. The pyramidals are found by dividing the products $1\times2\times3$, $2\times3\times4$, $3\times4\times5$, $4\times5\times6$, *&c.* each by $1\times2\times3$. In general, the figurate numbers of any order denoted by *m* are found by ſubſtituting ſucceſſively 1, 2, 3, 4, 5, *&c.* in place of x in the general expreſſion $\frac{x\,.\,x+1\,.\,x+2\,.\,x+3\,.\,\&c.}{1\,.\,2\,.\,3\,.\,4\,.\,\&c.}$ where the factors in the numerator and denominator are ſuppoſed to be multiplied by each other, and to be continued till the number in each be equal to that which expreſſes the order of the figurates required diminiſhed by unit. And when a figurate number of any order is divided by the correſponding figurate of any higher order, the numerator of the quotient is invariable, and x is in its denominator of as many dimenſions as there are units in the difference of the numbers that denote thoſe orders.

we may by it approximate to the value of the latter. Let AB represent unit; then the sum of the rectangles FB, EC, KH, LI, &c. and the sum of the ordinates AF, BE, CK, HL, &c. shall be expressed by the same number. But because the curve is convex towards the asymptote, the sum of those rectangles always exceeds the curvilineal area over the same base, by the triangles FEQ, EKS, KLT, LMZ, &c. and by the curvilineal spaces FuE, ExK, KzL, LyM, &c. The sum of those triangles approaches to one half of the rectangle FB, as its limit; therefore the sum of the rectangles FB, EC, KH, LI, &c. is greater than the curvilineal area AFEKLMI that is over the same base added to one half of the rectangle FB, the excess being the sum of the spaces FuE, ExK, KzL, LyM, &c. that are bounded by the arches FuE, ExK, KzL, &c. and their chords FE, EK, KL, &c. Hence, when AF is a term at a great distance from the beginning of a progression of this kind, the number that expresses the limit of the curvilineal area APNF added to one half of the term AF gives nearly the value of AF and the subsequent terms BE, CK, &c. the spaces FuE, ExK, &c. being neglected. But we may approximate to the value of such a progression more accurately in the following manner, that is deduced from art. 255. & 296. and will appear fully afterwards. Let the limit of the area APNF be expressed by A, the ordinate AF by a, the first fluxion of AF (the fluxion of the base being measured by AB, or unit) by b, the third, fifth and the subsequent fluxions of AF taken alternately, and always positively, by d, f, &c. Then the sum of the progression represented by AF, BE, CK, HL, &c. shall be found nearly by computing $A + \frac{1}{2}a + \frac{1}{12}b - \frac{1}{720}d + \frac{1}{30240}f + \&c.$ Or, if AR be taken towards b equal to one half of AB, the ordinate at R meet the curve in V, the limit of the area RPNV (or its value when RP is supposed to be produced infinitely) be now expressed by A, the first, third, fifth and subsequent fluxions of RV taken alternately, and always positively, by b, d, f, &c. then the sum of the terms AF, BE, CK, HL, &c. may be found by computing $A - \frac{1}{24}b + \frac{7}{5760}d - \frac{31}{967680}f + \&c.$ When it is not the limit of the progression that is required, but the sum of any number of terms of which AF is the first and af the last; then we may approximate to this sum by the first series if we sup-

ſuppoſe A to repreſent now the curvilineal area A*af*F, *a* the difference of AF and *af*, *b* the difference of their firſt fluxions, *d* the difference of their third fluxions, and ſo on. Or, if from *a* towards A we take *ar* equal to one half of AB, and the ordinate *rv* meet the curve in *v*, we may make uſe of the ſecond ſeries, provided A repreſent in it the area R*rv*V, *b* repreſent the difference of the fluxions of RV and *rv*, *d* the difference of their third fluxions, and ſo on.

353. When the limit of the progreſſion AF, BE, CK, HL, *&c.* is given, and the limit of the area is required, let the former be P; and, according to the firſt ſuppoſition, where *a* expreſſes AF, and *b*, *d*, *f*, *&c.* expreſs its firſt, third, fifth fluxions, *&c.* the limit of the area is found by computing $P - \frac{1}{2}a - \frac{1}{12}b + \frac{1}{720}d - \frac{1}{30240}f - \&c.$ But, according to the ſecond ſuppoſition, the limit of the area is found by computing $P + \frac{1}{24}b - \frac{7}{5760}d + \frac{31}{967680}f - \&c.$ The firſt expreſſion approximates to the area A*af*F when P is ſuppoſed to repreſent the ſum of the terms from AF to *af*, (excluding the latter,) *a* the difference of AF and *af*, *b* the difference of their firſt fluxions, *d* the difference of their third fluxions, and ſo on. And the ſecond ſeries approximates to the area R*rv*V when *a* is the difference of RV and *rv*, *b* the difference of their firſt fluxions, *d* the difference of their third fluxions, and ſo on. We refer the farther explication of this, with the demonſtration and examples, to the ſecond book. We ſhall now ſhew how progreſſions of fractions may be found at pleaſure that ſhall have aſſignable numbers equal to the limit of the ſum of the terms.

354. A ſeries of any number of quantities that continually decreaſe being given, their ſucceſſive differences form a new ſeries of terms, the ſum of which from the beginning is always equal to the exceſs of the firſt term of the firſt ſeries above its laſt term. Thus, if A, B, C, D, E, *&c.* be the terms of the firſt ſeries, it is manifeſt that the ſum of the differences of A and B, B and C, C and D, D and E, is the exceſs of A above E. If the terms of the firſt ſeries decreaſe in ſuch a manner that by continuing the progreſſion they may become leſs than any quantity how ſmall ſoever that can be aſſigned, (as the ordinates to the aſymptote become leſs than any line that may be given by

pro-

producing the figure,) then the firſt term of the firſt ſeries is the limit of the ſum of the ſecond ſeries. In like manner, the differences of the alternate terms of the firſt ſeries, as of A and C, B and D, C and E, &c. form a new progreſſion of terms, the ſum of any number of which is equal to the exceſs of the ſum of A and B, the firſt and ſecond term of the firſt ſeries, above the ſum of the laſt term and laſt but one; and the ſum of A and B is the limit of the ſum of the new ſeries. In general, if a progreſſion is formed by taking the differences of the firſt term A and the term whoſe place in the ſeries is expreſſed by any number n, of the ſecond term B and that whoſe place is $n+1$, of the third term C and that whoſe place is $n+2$, and ſo on; then the limit of the ſum of this new progreſſion ſhall be equal to the ſum of the terms A, B, C, D, &c. which preceed that term whoſe place is expreſſed by n. In this manner progreſſions may be found, at pleaſure, that may be continued without end, and have given numbers for the limits of their ſums: And this method differs not materially from that of the celebrated Mr. JAMES BERNOULLI.

355. For example, let the firſt ſeries be $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, &c. The ſucceſſive differences of thoſe terms are $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}$, &c. and the limit of the ſum of this progreſſion is unit, by the laſt article. If we multiply each term of this laſt ſeries by 2, (that the firſt term may be unit,) we ſhall have $1, \frac{1}{3}, \frac{1}{6}, \frac{1}{10}$, &c. which have the triangular numbers for their ſucceſſive denominators, unit being their common numerator; and therefore the limit of the ſum of this progreſſion is 2. The ſucceſſive differences of the terms of this latter ſeries being each multiplied by $\frac{1}{2}$, (that the firſt term of the new ſeries may be unit,) give $1, \frac{1}{4}, \frac{1}{10}, \frac{1}{20}$, &c. which have the pyramidal numbers for their ſucceſſive denominators; and the limit of the ſum of this progreſſion is $\frac{1}{2}$. In the ſame manner, the limit of the ſum of the fractions that have unit for their common numerator, and the figurate numbers of any order denoted by m for their ſucceſſive denominators, is found to be $\frac{m-1}{m-2}$.

356. The ſame ſeries $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, &c. being aſſumed again, the differences of the alternate terms are $\frac{2}{3}, \frac{2}{8}, \frac{2}{15}, \frac{2}{24}$, &c. the limit

limit of the ſum of which progreſſion is $1\frac{1}{2}$, by art. 354. and, dividing each term by 2, the limit of the ſum of $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, &c. is $\frac{1}{4}$. If we take the differences of the firſt term and that whoſe place is m, the ſecond term and that whoſe place is $m+1$, and ſo on; the common numerator of thoſe differences ſhall be $m-1$, and their ſucceſſive denominators ſhall be the products of 1 and m, 2 and $m+1$, 3 and $m+2$, and ſo on. The limit of the ſum of this progreſſion is the ſum of as many terms 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. as there are units in $m-1$, by art. 354. and if each term of that progreſſion be divided by $m-1$, that unit may become the common numerator, the terms $\frac{1}{m}$, $\frac{1}{2.m+1}$, $\frac{1}{3.m+2}$, $\frac{1}{4.m+3}$, &c. will ariſe, (where the factors in the denominators that are ſeparated by a point are ſuppoſed to be multipled by each other,) and the limit of the ſum of this progreſſion is equal to the ſum of the fractions 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. (continued till their number be $m-1$) divided by $m-1$. If we now aſſume the alternate terms only of the firſt ſeries beginning with the ſecond, that is, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, &c. and form a new ſeries by taking the difference of the firſt of theſe $\frac{1}{2}$ and that whoſe denominator is $m+1$, (m being any odd number,) of $\frac{1}{4}$ and that whoſe denominator is $m+3$, and ſo on; the new terms ſhall have $m-1$ for their common numerator, and the products of 2 and $m+1$, 4 and $m+3$, 6 and $m+5$, &c. for their ſucceſſive denominators: And the limit of the ſum of thoſe fractions ſhall be equal to the ſum of the fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, &c. that preceed $\frac{1}{m+1}$, the number of which fractions is $\frac{m-1}{2}$. By aſſuming the other alternate terms of the firſt ſeries, that is, 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, &c. and taking the differences of 1 and the term whoſe denominator is m, of $\frac{1}{3}$ and that whoſe denominator is $m+1$, and ſo on, the terms of the new ſeries ſhall have $m-1$ for their common numerator, and the products of 1 and m, 3 and $m+2$, 5 and $m+4$, &c. for their ſucceſſive denominators; and the ſum of the terms ſhall continually approximate to the ſum of the fractions 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, &c. that preceed $\frac{1}{m}$. We may likewiſe aſſume any other

equi-

equidiſtant terms of the firſt ſeries, and, by taking their differences, form new progreſſions the value of which ſhall be the firſt term that was aſſumed. If we aſſume the terms 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, &c. paſſing always over three terms, and divide the ſucceſſive differences of thoſe terms by 96, the ſeries will ariſe that is given by the celebrated Mr. MONMORT in Philoſ. Tranſact. n. 353. and is marked C; and therefore the value of this ſeries (when it is ſuppoſed to be continued infinitely) is $\frac{1}{96}$, by art. 354. If we aſſume the alternate terms of the firſt ſeries deduced in art. 355. *viz.* $\frac{1}{2}$, $\frac{1}{12}$, $\frac{1}{30}$, $\frac{1}{56}$, &c. and divide the ſucceſſive differences of thoſe terms by 2, the ſeries will ariſe that is marked B in the ſame Tranſaction; the value of which is therefore $\frac{1}{4}$. If we aſſume the firſt, fourth, ſeventh terms, &c. of the ſame ſeries from art. 355. paſſing always over two terms, their differences divided by 54 coincide with the ſeries marked A in the ſame place; and therefore the value of this ſeries is $\frac{1}{108}$. If we multiply the correſponding terms of any two progreſſions by each other, and if the products may become leſs than any given number by continuing them, (as for example, if we multiply the ſucceſſive terms of the ſeries 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. by the ſucceſſive powers of any fraction $\frac{1}{n}$,) their differences ſhall always give progreſſions of this kind. It is obvious from art. 354. how the ſum of any given number of terms is found in the progreſſions we have mentioned, or in any others that are deduced in this manner. But what follows ſeems to have a nearer relation to the method of fluxions, and to be of greater uſe.

357. Let us aſſume the ſeries $\frac{1}{m}$, $\frac{1}{m+1}$, $\frac{1}{m+2}$, $\frac{1}{m+3}$, &c. where unit is the common numerator, and the denominators increaſe by the continual addition of unit. The ſucceſſive differences of thoſe terms are $\frac{1}{m \cdot m+1}$, $\frac{1}{m+1 \cdot m+2}$, $\frac{1}{m+2 \cdot m+3}$, $\frac{1}{m+3 \cdot m+4}$, &c. and the limit of the ſum of this ſeries (or its value when it is ſuppoſed to be infinitely produced) is $\frac{1}{m}$, the firſt term of the ſeries which we aſſumed, by art. 354. The ſucceſſive differences of

of the terms of this laſt ſeries divided by 2 are $\frac{1}{m \cdot m+1 \cdot m+2}$, $\frac{1}{m+1 \cdot m+2 \cdot m+3}$, $\frac{1}{m+2 \cdot m+3 \cdot m+4}$, &c. and the limit of their ſum is therefore $\frac{1}{2m \cdot m+1}$, by the ſame article. The ſucceſſive differences of thoſe terms divided each by 3 are $\frac{1}{m \cdot m+1 \cdot m+2 \cdot m+3}$, $\frac{1}{m+1 \cdot m+2 \cdot m+3 \cdot m+4}$, &c. and the limit of their ſum is therefore $\frac{1}{3m \cdot m+1 \cdot m+2}$. The terms in each of thoſe progreſſions are formed from the firſt term by ſubſtituting ſucceſſively in its denominator $m+1$, $m+2$, $m+3$, $m+4$, &c. in place of m: And it appears, that, in general, if we ſubſtitute ſucceſſively m, $m+1$, $m+2$, $m+3$, &c. in place of m in the fraction $\frac{1}{m \cdot m+1 \cdot m+2 \cdot \&c.}$, (where the factors in the denominator are ſuppoſed to be continued till their number be $n+1$,) the terms that ſhall be produced in this manner ſhall form a progreſſion the value of which is found by ſubſtituting n, in the denominator of that fraction, in place of the laſt and greateſt factor. This uſeful theorem has been demonſtrated by ſeveral eminent Mathematicians, Dr. Taylor, Mr. Nicole, and of late by Mr. Stirling *, who has much improved the methods of approximating to the values of progreſſions that ariſe in the reſolution of problems. It is obvious from art. 354. how the ſum of any given number of terms may be found in the progreſſions we have deſcribed.

358. The ſame ſeries $\frac{1}{m}$, $\frac{1}{m+1}$, $\frac{1}{m+2}$, $\frac{1}{m+3}$, - - - $\frac{1}{m+n}$, &c. being aſſumed, where m is ſuppoſed to repreſent any quantity at pleaſure, and n any integer number; let a new progreſſion be formed by taking the differences of the firſt term $\frac{1}{m}$ and of $\frac{1}{m+n}$, of the ſecond $\frac{1}{m+1}$ and $\frac{1}{m+n+1}$, of the third $\frac{1}{m+2}$ and

* De ſummatione ſerierum, prop. 2.

$\frac{1}{m+n+2}$, and so on. Because the difference of the denominators of those fractions is always n, it is manifest that the terms of the new progression shall be $\frac{n}{m \cdot m+n}$, $\frac{n}{m+1 \cdot m+n+1}$, $\frac{n}{m+2 \cdot m+n+2}$, &c. The value of this progression is found by summing up all the terms in the first series that preceed the term $\frac{1}{m+n}$, by art. 354. the number of which terms is n. Hence it appears, that when n is any integer number, if we substitute successively m, $m+1$, $m+2$, $m+3$, &c. in place of m in the fraction $\frac{1}{m \cdot m+n}$, the value of the progression that shall be thus formed is equal to $\frac{1}{n \cdot m} + \frac{1}{n \cdot m+1} + \frac{1}{n \cdot m+2} +$ &c. those terms being continued till their number be n. But when n is a fraction, the value of the progression that is formed in the same manner cannot be assigned accurately in numbers, but we may approximate to it readily by a method explained below, (art. 361.) Several eminent Mathematicians have treated of this subject, besides those already mentioned, as Mr. Leibnitz, Messr. Bernoulli, and Mr. de Moivre; and various methods have been given by which an infinite variety of such progressions may be found. The following perhaps may be worth mentioning.

Fig. 147. 359. The right line AB being given, let the point P set out from A towards B, and proceed always in the same direction with an uniform motion; let the point p set out at the same time from B towards A with a velocity greater than that of A; and let it be constantly reflected with an uniform motion betwixt P and the fixed point B, so as to describe BD while P describes AD, and to describe DB + BE, EB + BF, FB + BG, &c. in the same times that P describes DE, EF, FG, &c. respectively: Then the spaces described by P and p betwixt the terms when they meet each other shall form two geometrical progressions, and the common ratio of the terms in both shall be that of the sum of the velocities of P and p to their difference.

rence. If we ſuppoſe the velocity of p to increaſe or decreaſe every time it comes to B, (but ſtill ſo as to be greater than the conſtant velocity of P,) and to continue uniform till it return again to B, the ſpaces deſcribed by P betwixt the terms when it meets p ſhall form progreſſions of various kinds, that are eaſily determined when the rule is given according to which the velocity of p increaſes or decreaſes. But to obtain progreſſions whoſe terms may have more ſimple expreſſions, let us conceive p not to be reflected from P to B, but every time it meets P to be inſtantly brought back to B, and to ſet out anew from B towards A till it meet P again, ſo as to deſcribe BD, BE, BF, BG, *&c.* in the ſame times, reſpectively, that P deſcribes AD, DE, EF, FG, *&c.* Then the ſpaces deſcribed by P ſhall form a progreſſion of terms that may be continued without end, when the velocities of P and p (however variable they may be) are always in an aſſignable ratio to each other while P deſcribes AB; the ſum of thoſe terms is always leſs than AB, but approaches to it as its limit while the progreſſion is continued. Suppoſe the motion of P to be invariable, and the motion of p to be uniform while P deſcribes any one term; let the conſtant velocity of P be expreſſed by m; and the velocities of p, while P deſcribes any two ſucceſſive terms EF, FG, be expreſſed by V and u, reſpectively: then ſhall FG be to EF as V is to $m+u$, becauſe FG is to BF as m is to $m+u$, and BF is to EF as V is to m. The ſum of any number of terms deſcribed by P, as AG, is found by ſubducting from AB a right line that is to the laſt term FG as u is to m; for ſuch a line is equal to BG.

360. The velocity of P being expreſſed by m, (as in the laſt article,) let the ſucceſſive velocities of p increaſe equally, and be expreſſed by $n+1$, $n+2$, $n+3$, $n+4$, *&c.* and let r be equal to $m+n$. Then, ſince AD the firſt term deſcribed by P is to AB as m is to $r+1$, if AB repreſent unit, AD ſhall be expreſſed by $\frac{m}{r+1}$. The ratio of V to $m+u$ is ſucceſſively that of $n+1$ to $r+2$, $n+2$ to $r+3$, $n+3$ to $r+4$, *&c.* Therefore, ſuppoſing (after Sir ISAAC NEWTON's manner) A to expreſs the firſt term $\frac{m}{r+1}$, B the ſecond term, C the third, and

so on, the terms described by P are $\frac{m}{r+1}$, $\frac{A.n+1}{r+2}$, $\frac{B.n+2}{r+3}$, $\frac{C.n+3}{r+4}$, &c. The limit of the sum of this progression is AB, or unit; and the sum of any number of terms denoted by z, the last term being l, is $1 - \frac{l.z+n}{m}$. For example, if we suppose the first velocity of p to be equal to the constant velocity of P, and the successive velocities of p to be 1, 2, 3, 4, &c. that is, if m be unit, and n vanish, the terms described by p shall be $\frac{1}{1.2}$, $\frac{1}{2.3}$, $\frac{1}{3.4}$, $\frac{1}{4.5}$, &c. the sum of which must be always less than unit, as in art. 355. Any term FG whose place in the series is denoted by z is $\frac{1}{z.z+1}$; and therefore AG, the sum of as many terms from the beginning as there are units in z, is (by what was shewn in the last article) $\frac{z}{z+1}$; and the sum of the same number of fractions, that have unit for their common numerator, and the triangular numbers 1, 3, 6, 10, &c. for their successive denominators, is 2AG, or $\frac{2z}{z+1}$. The rest remaining, if m be equal to 2, the terms described by P shall be $\frac{4}{1.2.3}$, $\frac{4}{2.3.4}$, $\frac{4}{3.4.5}$, &c. FG is expressed by $\frac{4}{z.z+1.z+2}$, AG by $1 - \frac{2}{z+1.z+2}$; and the sum of the same number of fractions, that have unit for their common numerator, and the pyramidal numbers for their successive denominators, is $\frac{3}{2} - \frac{3}{z+1.z+2}$. In the same manner it will appear, that, the successive velocities of p being represented by 1, 2, 3, 4, &c. if the constant velocity of P be expressed by any positive integer number m, the sum of the same number of fractions that have unit for their common numerator, and the figurate numbers of the order $m+2$ for their successive denominators, shall be to AG as $m+1$ is to m, and is therefore equal to $\frac{1}{m}$ multiplied by the

excess

excels of $m+1$ above the fraction $\frac{2 \cdot 3 \cdot 4 \cdot \&c.}{x+1 \cdot x+2 \cdot x+3 \cdot \&c.}$ where the factors are supposed to be continued in the numerator and denominator till the number in each be m; as has been demonstrated by Mr. BERNOULLI *de seriebus infinitis*, § 18. & 19.

361. Supposing n still to vanish, or m to be equal to r, the terms described by P are $\frac{m}{m+1}$, $\frac{m \cdot m+1}{m+1 \cdot m+2}$, $\frac{2 \cdot m \cdot m+1}{m+1 \cdot m+2 \cdot m+3}$, $\frac{2 \cdot 3 \cdot m}{m+1 \cdot m+2 \cdot m+3 \cdot m+4}$, &c. and since the limit of their sum is unit, it follows (dividing by mm) that $\frac{1}{mm}$ is the limit of the sum of $\frac{1}{m \cdot m+1}$, $\frac{1}{m \cdot m+1 \cdot m+2}$, $\frac{1 \cdot 2}{m \cdot m+1 \cdot m+2 \cdot m+3}$, $\frac{1 \cdot 2 \cdot 3}{m \cdot m+1 \cdot m+2 \cdot m+3 \cdot m+4}$, &c. From which, and what was shewn in art. 357. it follows, that we approximate to the sum of the terms which arise when we substitute successively m, $m+1$, $m+2$, $m+3$, &c. in place of m in the fraction $\frac{1}{mm}$, by summing up the series $\frac{1}{m^2}$, $\frac{1}{2m \cdot m+1}$, $\frac{2}{3m \cdot m+1 \cdot m+2}$, $\frac{2 \cdot 3}{4m \cdot m+1 \cdot m+2 \cdot m+3}$, &c. or $\frac{P}{m^2}$, $\frac{A}{2 \cdot m+1}$, $\frac{4B}{3 \cdot m+2}$, $\frac{9C}{4 \cdot m+3}$, &c. where A represents the first term $\frac{P}{m}$, B the second term, and so on. Of the use of this, see Mr. STIRLING's treatise *de summatione serierum*, p. 28. In general, when n does not vanish, we found that unit is the limit of the sum of the progression $\frac{m}{r+1}$, $\frac{A \cdot n+1}{r+2}$, $\frac{B \cdot n+2}{r+3}$, $\frac{C \cdot n+3}{r+4}$, &c. Therefore, dividing by m and r, $\frac{P}{m \cdot r}$ is the value of the progression $\frac{1}{r \cdot r+1}$, $\frac{n+1}{r \cdot r+1 \cdot r+2}$, $\frac{n+1 \cdot n+2}{r \cdot r+1 \cdot r+2 \cdot r+3}$, &c. And, by art. 357. we shall appproximate to the sum of the terms that arise by substituting successively r, $r+1$, $r+2$, &c. in place of r in the fraction $\frac{1}{r \cdot m}$, or $\frac{1}{r \cdot r-n}$, if we sum up the

terms

terms $\frac{1}{r}$ $\frac{n+1}{2r.r+1}$ $\frac{n+1.n+2}{3r.r+1.r+2}$, &c. or the ſeries $\frac{1}{r} + \frac{A.n+1}{2.r+1}$ $+\frac{2B.n+2}{3.r+2} + \frac{3C.n+3}{4.r+3} +$ &c. When n is any negative fraction equal to $-f$, $\frac{1}{r.r-f}$ becomes $\frac{1}{r.r+f}$, and the ſeries becomes $\frac{1}{r} + \frac{A.1-f}{2.r+1} + \frac{2B.2-f}{3.r+2} + \frac{3C.3-f}{4.r+3} +$ &c. which therefore approximates continually to the ſum of the terms that ariſe when r, $r+1$, $r+2$, &c. are ſubſtituted ſucceſſively for r in $\frac{1}{r.r+f}$; and this agrees with what is ſhewn in a different manner in the ſame excellent treatiſe, p. 12. & 26. where this ſeries is applied for finding readily the value of the progreſſion $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4}$ $+ \frac{1}{4.5} + \frac{1}{5.6} +$ &c. that was invented by Lord BROUNKER for the quadrature of the hyperbola.

FIG. 148. 362. Let the point P move now in the right line BA produced beyond A, and the velocity of p be always greater than that of P, that it may overtake it; let the motion of P be ſtill uniform, and its velocity be repreſented by m; and let the ſucceſſive velocities of p by which it deſcribes any two terms BF, BG (while P deſcribes EF and FG) be V and u, reſpectively. Then ſhall FG be to EF as V is to $u - m$, becauſe FG is to BF as m is to $u - m$, and BF to EF as V is to m. The ſum of any ſpaces AD, DE, EF, FG, deſcribed by P, added to BA, (which is ſuppoſed to repreſent unit,) is BG, which is to FG the laſt of thoſe ſpaces as u is to m; and if FG the laſt term of the progreſſion be expreſſed by l, then BG (the ſum of the terms including AB) ſhall be $\frac{lu}{m}$. When the motion of p is likewiſe uniform, the ſpaces deſcribed by P continually increaſe in a geometrical progreſſion; the common ratio of the terms is that of the velocity of p to its exceſs above the velocity of P; AG the ſum of any number of terms deſcribed by P is found by ſubducting AB (or unit) from BG, which is to the laſt term FG, as the velocity of p is to the velocity of P, or as the number

that

that expreſſes the common ratio of the terms is to the ſame number diminiſhed by unit: And hence the common rule for finding the ſum of any number of terms in a geometrical progreſſion may be deduced. When the conſtant velocity of P is denoted by m, and the ſucceſſive velocities of p by $r+1$, $r+2$, $r+3$, &c. let n be equal to $r-m$, and the terms deſcribed by P ſhall be $\frac{m}{n+1}$, $\frac{A.r+1}{n+2}$, $\frac{B.r+2}{n+3}$, &c. where A repreſents the firſt term, B the ſecond, and ſo on. And if we prefix AB, or unit, to thoſe terms, the ſum of any number from the beginning to the term whoſe place is denoted by z ſhall be to this term as $z+r-1$ is to m. When m is an integer poſitive number, and r is equal to m, (or n vaniſhes,) theſe terms are the figurate numbers of the order denoted by m, the ſecond number of each order (or the firſt ſpace deſcribed by P) being m. In this caſe, when m is unit, each term is equal to AB, or unit: When m is 2, the terms are in arithmetical progreſſion, the laſt term is z, and the ſum of the terms $\frac{z.z+1}{2}$: When m is 3, the terms are the triangular numbers, the laſt term is $\frac{z.z+1}{2}$, and the ſum of the terms is $\frac{z.z+1.z+2}{2.3}$. And, in general, the ſum of the figurate numbers of any order m from the firſt, or unit, to the laſt, (whoſe place is ſuppoſed to be denoted by z,) is found by multiplying this laſt by $\frac{z+m-1}{m}$, or by computing $\frac{z.z+1.z+2.z+3.\text{ &c.}}{1.2.3.4.\text{ &c.}}$, where the factors are ſuppoſed to be continued in the numerator and denominator till there be as many in each as there are units in m. We might ſuppoſe the velocity of p to obſerve other rules, or the velocity of P to vary likewiſe; but, not to inſiſt further on this ſubject here, we ſhall only add, that there are other methods beſides that deſcribed in art. 350. & 351. by which it may be known when a progreſſion of fractions has a limit, or not; ſuch is the rule given by an author we have often mentioned, That when A, B, C are any ſucceſſive terms of the progreſſion at a ſufficient diſtance

ſtance from the beginning, and the ratio of A to C is leſs than the ratio of the difference of A and B to the difference of B and C, the progreſſion may be continued till its ſum exceed any given number; but when it is otherwiſe, the ſum of the progreſſion has a limit.

CHAP. XI.

Of the Curvature of Lines, its Variation, and the different kinds of Contact of the Curve and Circle of Curvature, the Cauſtics by Reflexion and Refraction, the centripetal Forces, and other Problems that have a dependence upon the Curvature of Lines.

363. ANy two right lines applied upon each other, perfectly coincide; and the rectitude of lines admits of no variety. Arches of equal circles applied upon each other, perfectly coincide likewiſe; and the curvature is uniform in all the parts of the ſame, or of equal circles. Arches of unequal circles cannot be applied upon each other ſo as to coincide; but when they touch each other, the arch of the greater circle is leſs inflected from the common tangent, and paſſes betwixt it and the arch of the leſſer circle through the angle of contact formed by them, and is therefore leſs curve. Any two arches of curve lines touch each other when the ſame right line is the tangent of both at the ſame point; but when they are applied upon each other in this manner, they never perfectly coincide, unleſs they are ſimilar arches of equal and ſimilar figures: and the curvature of lines admits of indefinite variety. Becauſe the curvature is uniform in a given circle, and may be varied at pleaſure in them by enlarging or diminiſhing their diameters, the flexure or curvature of circles ſerves for meaſuring that of other lines. There is but one right line that can be the tangent of a given arch of a curve at the ſame point; but circles of an indefinite variety touch it there; and theſe have various degrees of more and leſs intimate contact with it.

364. As

364. As of all the right lines that can be drawn through a given point in the arch of a curve, that is the *tangent* which touches the arch ſo cloſely that no right line can be drawn between them, (art. 181.) ſo of all the circles that touch a curve in any given point, that is ſaid to have the *ſame curvature* with it, which touches it ſo cloſely that no circle can be drawn through the point of contact between them, all other circles paſſing either within or without them both. This circle is called the *circle of curvature*, its center the *center of curvature*, and its ſemidiameter the *ray of curvature*, belonging to the point of contact. The arch of this circle cannot coincide with the arch of the curve, but it is ſufficient to denote it the circle of curvature that no other circle can paſs between them; as the tangent of the arch of a curve cannot coincide with it, but is applied to it ſo that no right line can be drawn between them. As in all figures, rectilineal ones excepted, the poſition of the tangent is continually varying; ſo the curvature is continually varying in all curvilineal figures, the circle only excepted. As the curve is ſeparated from its tangent by its flexure or curvature, ſo it is ſeparated from the circle of curvature in conſequence of the increaſe or decreaſe of its curvature: and as its curvature is the greater or leſs according as it is more or leſs inflected from the tangent, ſo the variation of curvature is the greater or leſs according as it is more or leſs ſeparated from the circle of curvature. It is manifeſt, that there is but one circle of curvature belonging to an arch of a curve at the ſame point; for if there were two ſuch circles, any circles deſcribed between theſe through that point would paſs between the curve and circle of curvature, againſt the ſuppoſition.

365. When any two curve lines touch each other in ſuch a manner that no circle can paſs between them, they muſt have the ſame curvature, by the laſt article; for the circle that touches the one ſo cloſely that no circle can paſs between them, muſt touch the other in the ſame manner. It will appear from the following propoſition, that circles may touch curve lines in this manner, that there may be indefinite degrees of more or leſs intimate contact between the curve and the circle of curvature, and that a conic ſection may be deſcribed that ſhall have the

same curvature with a given line at a given point, and the same variation of curvature, or a contact of the same kind with the circle of curvature.

PROP. XXXII.

366. *Let any curve* EMH *and a circle* ERB *touch the right line* ET *on the same side at* E; *let any right line* TK *parallel to the chord* EB *meet the tangent in* T, EMH *in* M, *and a curve* BKF *that passes through* B *in* K. *Then, if the rectangle* MTK *be always equal to the square of* ET, *the curvature of* EMH *at* E *shall be the same as that of the circle* ERB; *and the contact of* EM *and* ER *shall be always the closer the less the angle is that is contained at* B *by the curve* BKF *and the circle of curvature* BQE.

FIG. 149. & 150.

Let the right line TK meet the circle in R and in Q, and, since the rectangle RTQ is equal to the square of ET, it must be equal to the rectangle MTK, by the supposition; and therefore RT is to MT as TK is to TQ. Suppose first that BK the part of the curve BKF that is next to the point B, adjoining to it, falls without the circle BQ; and suppose the right line TK by moving parallel to itself to approach to EB till it coincide with it; and while the point K describes KB, TK being greater than TQ, RT must be greater than MT, and the arch EM of the curve must pass without the circle ER betwixt it and the tangent ET: And since any circle described through E, upon a chord less than EB touching ET, falls within the circle ERB, it is manifest that no such circle can pass betwixt the curve EM and circle ERB. Let any circle *Erb*, described upon a chord *Eb* greater than EB, touch ET, and meet TK in *r* and *q*; and, since the rectangle *rTq* is equal to the square of ET, or the rectangle MTK, MT is to *r*T as *Tq* is to TK: And, since the curve EKB passes through B (by the supposition) so that the part of it that is next adjoining to B must be within the arch *bq* of the circle *bq*E, it follows that while K describes

FIG. 149.

this

this part of FKB, Tq muſt be greater than TK, and, conſequently, MT greater than rT. Therefore the arch Er of the circle Erb is without the curve EM, and paſſes betwixt it and the tangent ET. Therefore no circle whatever can paſs betwixt EM the arch of the curve and ER the arch of the circle; and, conſequently, the circle ERB has the ſame curvature with EM at E. Suppoſe now that the part of the curve BKF that is next adjoining to B falls within BQ the arch of the circle BQE; then, while K deſcribes this part of the curve FKB, TK being leſs than TQ, RT muſt be leſs than MT, and the arch EM of the curve muſt fall within ER the arch of the circle; and, ſince any circle deſcribed through E upon a chord greater than EB falls without the circle ER, it is manifeſt that no ſuch circle can paſs betwixt ER and EM. Nor can any circle Erb, deſcribed upon a chord Eb leſs than EB, touching ET, paſs between ER and EM: for let TK meet this circle in r and q, and MT being to rT as Tq is to TK, and Tq being leſs than TK while K deſcribes the part of the curve FKB that is next adjoining to B, MT muſt be leſs than rT; and, conſequently, the arch Er of the circle Erb muſt fall within EM the arch of the curve. Therefore, in either caſe, all the circles that can be deſcribed through E fall without both ER and EM, or within them both, and no circle whatever can paſs between them, when the rectangle MTK is always equal to the ſquare of ET, and the curve in which K is always found paſſes through B; that is, the circle ERB and the curve EM have the ſame curvature at E, by art. 364. which was the firſt part of the propoſition. Fig. 150.

367. Let Em any other curve touching ET in E, and fkB another curve paſſing through B, meet TK in m and k, and let the rectangle mTk be likewiſe equal to the ſquare of ET; then the curvature of Em at E ſhall be the ſame as that of the circle ERB, or that of the curve EM, by what has been demonſtrated. Becauſe the rectangles mTk, MTK, RTQ are equal to each other, Tm is to TM as TK is to Tk, and Tm to TR as TQ is to Tk. Therefore, if the arch Bk paſs between BK, the arch of the curve BKF, and BQ the arch of the circle BQE, the curve Em muſt paſs between EM, the arch of the curve Fig. 151.

EMH, and ER the arch of the circle of curvature ERB; ſo that *Em* muſt have a cloſer contact with this circle than EM has with it: And the leſs the angle is that is formed by the curve FKB and the circle of curvature EQB at B, the cloſer is the contact at E of the curve EMH and the circle of curvature ERB. Thus the curve BKF by its interſection with EB determines the curvature of EM, and by the angle in which it cuts the circle of curvature it determines the degree of contact of EM and that circle, the angle BET and right line ET being given.

368. Cor. I. It appears from the demonſtration, that according as the arch BK of the curve BKF falls without, or within, the arch BQ of the circle BQE, the arch EM of the curve EMH falls without, or within, the circle ERB; that when the curve FKB cuts the circle ERB in B, the curve HME cuts the circle of curvature in E; that when the curve FKB is on the ſame ſide of the circle BQE on both ſides of B, the curve HME continued on both ſides of E is on the ſame ſide of the circle of curvature; and that the contact of the curve EMH and the circle of curvature is cloſeſt when the curve BK touches the arch BQ in B, the angle BET being given, but it is fartheſt from this, or is moſt open, when BK touches the right line EB in B.

369. Cor. II. There may be indefinite degrees of more and more intimate contact between a circle ERB and a curve EMH. The firſt degree is, when the ſame right line touches them both in the ſame point; and a contact of this ſort may take place betwixt any circle and any arch of any curve. The ſecond is, when the curve EMH and circle ERB have the ſame curvature, and the tangents of the curve BKF and circle BQE interſect each other at B in any aſſignable angle. The contact of the curve EM and circle of curvature ER at E is of the third degree, or order, and their oſculation is of the ſecond, when the curve BKF touches the circle BQE at B, but ſo as not to have the ſame curvature with it. The contact is of the fourth degree, or order, and their oſculation of the third, when the curve BKF has the ſame curvature with the circle BQE at B, but ſo as that their contact is only of the ſecond degree: And this grada-

gradation of more and more intimate contact, or of approximation towards coincidence, may be continued indefinitely, the contact of EM and EK at E being always of an order two degrees closer than that of BK and BQ at B. There is however an indefinite variety comprehended under each order. Thus, when EM and ER have the same curvature, the angle formed by the tangents of BK and BQ admits of indefinite variety, and the contact of EM and ER is the closer the less that angle is: And when that angle is of the same magnitude, the contact of EM and ER is the closer the greater the circle of curvature is: for since TR is to TM as TK is to TQ, RM (which subtends the angle of contact MER) is to TR as KQ is to TK; and, consequently, RM is to KQ as the square of ET is to the rectangle KTQ; so that, when ET is given, RM is as KQ directly and the rectangle KTQ inversely; and when KQ is given, RM is less in proportion as the rectangle KTQ is greater. When BK touches the circle BQ at B, they may touch it on the same, or on different sides of their common tangent; and the angle of contact KBQ may admit of the same variety with the angle of contact MER in the former case. But as there is seldom occasion for considering those higher degrees of more intimate contact of the curve EMH and circle of curvature ERB, we shall call their *contact* or *osculation of the same kind*, when, the chord EB and angle BET being given, the angle contained by the tangents of BK and BQ is of the same magnitude.

369. Cor. III. The curvature is uniform in the circle only. When the curvature of EMH increases from E towards H, and consequently corresponds to that of a circle gradually less and less, the arch EM falls within ER the arch of the circle of curvature, and BK is within BQ. When the curvature of EM decreases from E towards H, and consequently corresponds to that of a circle that is gradually greater and greater, the arch EM falls without ER the arch of the circle of curvature; and BK is without BQ. According as the curvature of EM varies more or less, it is more or less unlike to the uniform curvature of a circle; the arch of the curve EMH separates more or less from the arch of the circle of curvature ERB, and the angle con-

contained by the tangents of BKF and BQE at B is greater or leſs. Thus the *quality* of curvature (as it is called by Sir ISAAC NEWTON in a treatiſe lately publiſhed by the ingenious Mr. COLSON) depends on the angle contained by the tangents of BK and BQ at B; and the meaſure of the inequability or variation of curvature is as the tangent of this angle, the radius being given, and the angle BET being right; for the index of this variation is as the fluxion of the ray of curvature directly and the fluxion of the curve inverſely: and we ſhall ſhew afterwards (art. 386.) that the fluxion of TK when M comes to E is always to the fluxion of the ray of curvature in the invariable ratio of two to three. The meaſure of the angle of contact MER, contained by the curve and circle of curvature, depends not on the angle in which BK interſects BQ, only; but is as the tangent of this angle directly and the ſquare of the ray of curvature inverſely (as will appear afterwards) when the tangents of BK and BQ interſect each other in any aſſignable angle; and this meaſure obſerves other proportions when BK touches BQ at B.

FIG. 152. 370. COR. IV. The rays of curvature of ſimilar arches in ſimilar figures are in the ſame ratio as any homologous lines of theſe figures, and the curve BK cuts BQ in the ſame angle, or the variation of curvature is the ſame. For let EM and *em* be any ſimilar arches ſituated as we deſcribed in art. 122. ſo that, S being a given point, if SM meet EM and *em* in M and *m*, SM may be always to S*m* in the ſame ratio as SE is to S*e*. Let SM and S*m* meet the tangents ET, *et* (which are parallel to each other becauſe the figures are ſimilar and ſimilarly ſituated) in Z and *z*; and MZ ſhall be to *mz*, TM to *tm*, and ET to *et*, as SE is to S*e*. Therefore, ſince the rectangle MTK is to *mtk* as the ſquare of ET is to the ſquare of *et*, TK is to *tk*, MK to *mk*, and SK to S*k*, as SE is to S*e*; and the figures SKB, S*kb* are likewiſe ſimilar by art. 122. Therefore EB is to *eb* as SE is to S*e*; and (becauſe the angles BET, *bet* are equal) the rays of curvature are in the ſame ratio. The tangents of BK and *bk* at B and *b* are parallel, and cut the circles of curvature BQ and *bq* in equal angles: Therefore the variation of curvature is the ſame in ſimilar arches. When the tangent of BK cuts the tangent of

Pl. XVI. Pa. 310.

Fig. 142. N. 1

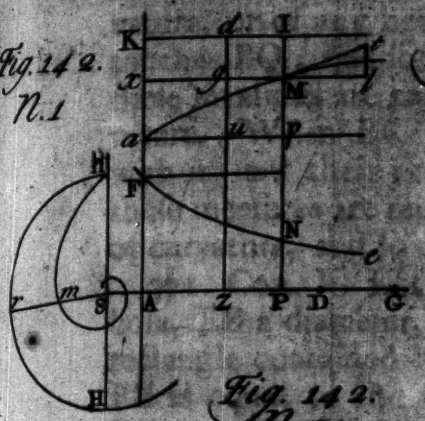


Fig. 143.

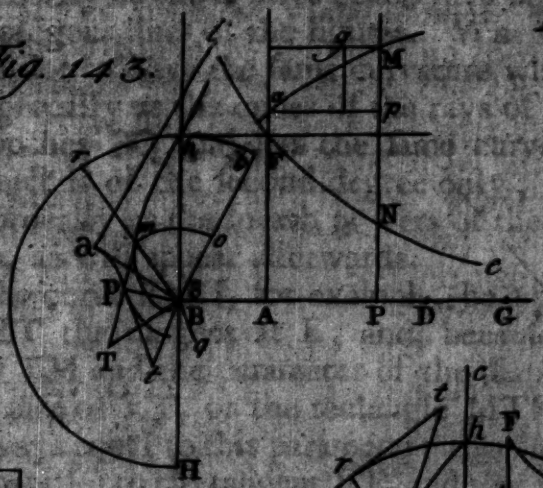


Fig. 144.

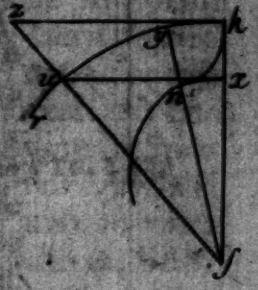


Fig. 142. N. 2

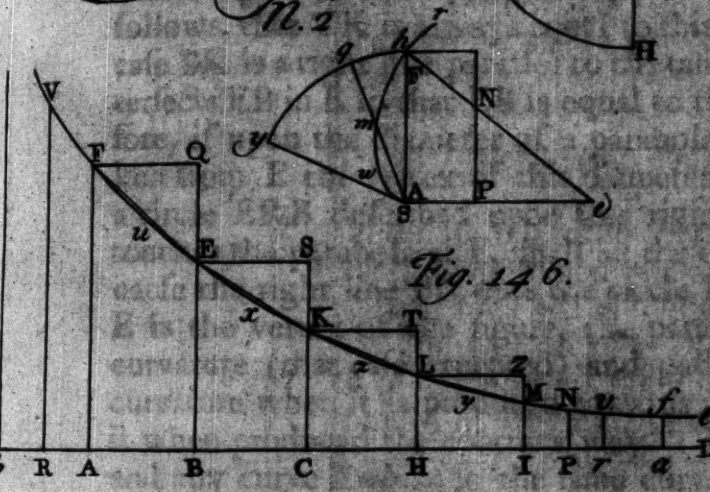


Fig. 145.

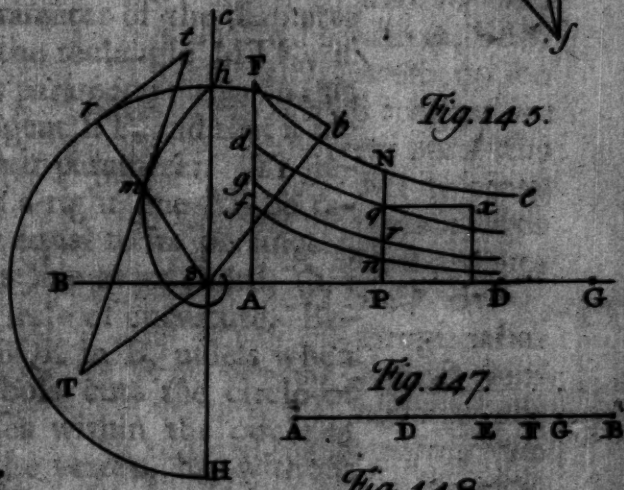


Fig. 146.

Fig. 147.

Fig. 148.

Fig. 149.

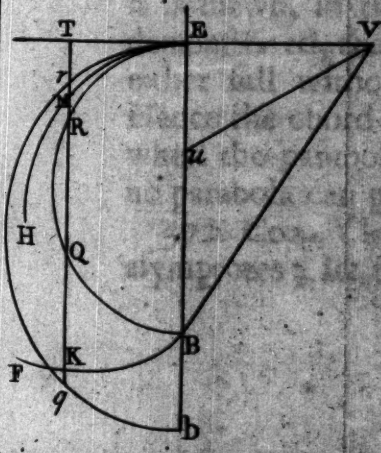


Fig. 150.

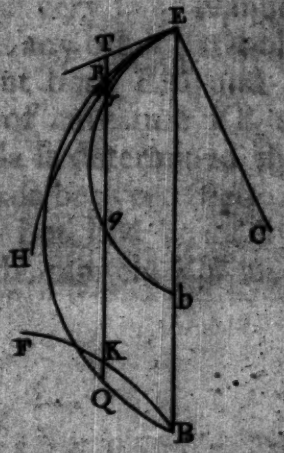


Fig. 151.

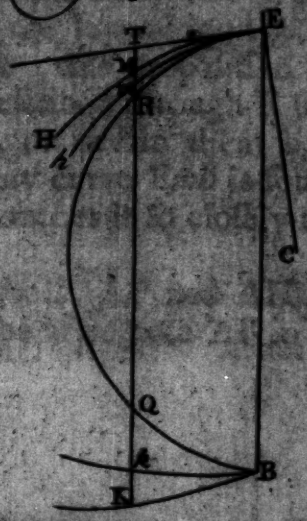
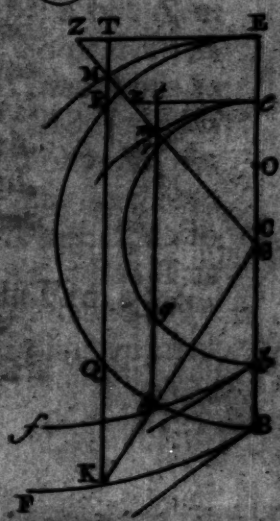
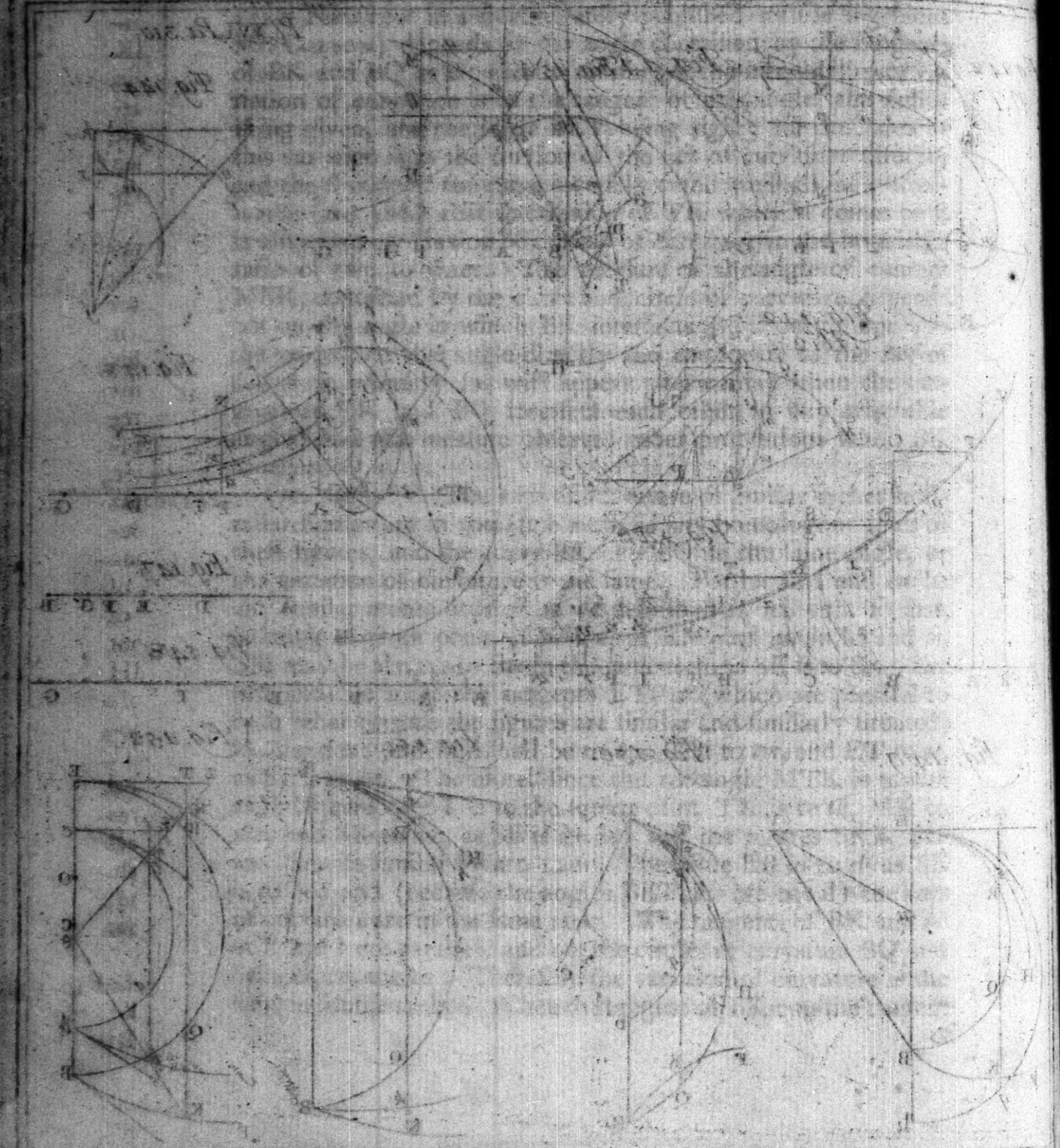


Fig. 152.





of BQ in any aſſignable angle, the meaſures of the angles of contact MER and *mer* contained by ſimilar arches and their circles of curvature, are reciprocally as the ſquares of the rays of curvature, or of any homologous lines of the figures. When BK touches BQ, but ſo as not to have the ſame curvature with it, thoſe meaſures are reciprocally as the cubes of the rays of curvature. When BK touches BQ and has the ſame curvature with it, and their contact is of the ſecond degree only; then thoſe meaſures are reciprocally as the fourth powers of the rays of curvature, and ſo on, as will appear afterwards.

371. Cor. V. Let the curve EMH, for example, be a parabola, EB a diameter, ET the tangent at E; and, becauſe the rectangle contained by TM and the parameter of the diameter EB is equal to the ſquare of ET, or the rectangle MTK, it follows, that TK is always equal to this parameter; that in this caſe BK is a right line parallel to the tangent ET, and that it interſects EB in B ſo that EB is equal to that parameter. Therefore, if upon the diameter of a parabola a right line EB be taken from E the vertex of this diameter equal to its parameter, a circle ERB deſcribed upon this right line as its chord, that touches the parabola at E, ſhall be the circle of curvature. Becauſe the right line BK cuts the circle BQE in B, unleſs when E is the vertex of the figure, the parabola cuts the circle of curvature (that caſe excepted) and paſſes within the circle of curvature when it is produced towards the vertex, but without it when produced the contrary way. When a parabola EMH and any curve *Emb* have the ſame curvature at E, no parabola can be drawn through E betwixt EMH and *Emb*; for ſuppoſing the rectangle *m*T*k* to be always equal to the ſquare of ET, the curve *fk* ſhall paſs through B, (by the ſuppoſition:) and it is ſhewn, in the very ſame manner that this propoſition was demonſtrated, that any other parabola deſcribed through E ſhall either fall without both EM and E*m*, or within them both. Hence the chord of curvature EB of any curve *Emb* is found, when the parabola is determined that touches it ſo cloſely that no parabola can paſs between them. Fig. 153.

372. Cor. VI. Let EM be an hyperbola, AF and AH the aſymptotes; let the tangent ET meet the aſymptote AF in V; Fig. 154.

let the angle EVB be made equal to the angle HAP contained by the aſymptotes, and VB meet EB parallel to the aſymptote within the curve in B: then a circle deſcribed upon the chord EB that touches ET ſhall be the circle of curvature. For, in this caſe, the point K is always found in the right line VB; becauſe, if MT produced meet EP parallel to AH in Z, and TG parallel to VB meet EB in G, then, becauſe of the ſimilar triangles EZT, ETG, the rectangle contained by EG and ZT ſhall be equal to the ſquare of ET, or the rectangle MTK; and TK ſhall be to EG as ZT is to TM, or (by the properties of the hyperbola) as PZ is to EZ, or VT to TE, or GB to EG; therefore TK is equal to GB, and K is always found in the right line VB. Becauſe VB cuts the circle BQ, unleſs when the angles EBV and BEV are equal and conſequently E is the vertex of the figure, it follows that the hyperbola cuts the circle of curvature in all other caſes. When ET the tangent of any curve EM and BV the tangent of the curve BK are given, an hyperbola may be deſcribed through E, that ſhall have the ſame curvature with EM and the ſame variation of curvature, or a contact with the circle of curvature of the ſame kind, by producing thoſe tangents till they meet in V, and deſcribing through E an hyperbola that has an aſymptote through V parallel to EB, and another aſymptote through *u* (E*u* being taken equal to EV upon VE produced beyond E) that conſtitutes with the former an angle *u*AV equal to EVB, or that makes the angle E*u*A equal to EBV.

FIG. 155. 373. COR. VII. Let EMH be any conic ſection, ET the tangent at E, HI a tangent parallel to EB that meets ET in I, and let EMH meet EB in G. Take EB to EG in the ſame ratio as the ſquare of EI is to the ſquare of HI, or (when the conic ſection has a center) as the ſquare of the ſemidiameter O*a* parallel to ET is to the ſquare of the ſemidiameter OA parallel to EB, and a circle deſcribed upon the chord EB that touches ET ſhall be the circle of curvature. For let TM meet the conic ſection again in *m*; and, ſince the rectangle MT*m* is to the ſquare of ET, or the rectangle MTK, as the ſquare of HI is to the ſquare of EI, T*m* is always to TK in the ſame ratio; and, conſequently, the point K is always found in a conic ſection,

ction, that meets EG in B ſo that EB is to EG as the ſquare of EI is to the ſquare of HI. Let GV the tangent of the conic ſection EMG at G meet ET in V; and, becauſe TV is to TK in an invariable ratio, and their fluxions are in the ſame ratio, it follows from prop. 14. that BV ſhall be the tangent of the conic ſection BKF at B. Hence it eaſily appears whether the arch EM of the conic ſection falls without or within the circle of curvature, and the different degrees of oſculation betwixt the conic ſection and the circle of the ſame curvature may be compared together. For let any conic ſection EMH meet the chord of curvature EB in G, draw the tangent GV meeting ET in V, join BV, and, the angle BET being given, the nearer the angle EBV is to BET, or the nearer the ratio of EV to BV is to a ratio of equality, the cloſer is the contact of the conic ſection EM and of the circle of curvature at E. Let the angle EB*v* be made equal to BET, and let B*v* meet ET in *v*, join G*v*, and of all the conic ſections which can be deſcribed through E and G having the ſame curvature at E with the circle ERB, that which touches the right line G*v* has the cloſeſt contact with the circle: When BET is a right angle, or EB is the diameter of the circle of curvature, EG is in this caſe the axis of the conic ſection (becauſe the angle EB*v* is a right one,) EB is the parameter of this axis, and when the points G and B are on the ſame ſide of E, EMG is an ellipſe, and EG is the greater or the leſſer axis according as EG is greater or leſs than EB.

374. Cor. VIII. The propoſitions relating to the curvature of the conic ſections, which are delivered by authors on this ſubject, follow from what has been demonſtrated. 1. When the chord of curvature EB paſſes through O the center of the conic ſection, A coincides with E, EG is a diameter, OA and O*a* are conjugate ſemidiameters, and, ſince EB is to EG as the ſquare of O*a* is to the ſquare of OA (or OE,) or as the parameter of the diameter EG is to EG, it follows, that EB the chord of the circle of curvature is equal to the parameter of EG the diameter that paſſes through the point of contact. This was ſhewn of the parabola, in art. 371. 2. Let C be the center of the circle of curvature, and C*b* be perpendicular on the dia- Fig. 156.

diameter EG in *b*; let O*a* meet EC in N: and, the triangles EON, EC*b* being similar, the rectangle CEN is equal to the rectangle OE*b*, which is equal to the square of O*a*, because E*b* is one half of the parameter of EG. Therefore the square of the semidiameter O*a* is to the rectangle contained by O*a* and EN, or the invariable rectangle contained by half the transverse and half the conjugate axis, as the ray of curvature CE is to O*a*; and the cube of the semidiameter O*a*, that is conjugate to OE which passes through the point of contact, is equal to the solid contained by the ray of curvature and that invariable rectangle, as is shewn by Mr. de Moivre, *Misc. analyt.* p. 235.

FIG. 157. 3. Let EH be an ordinate to an axis of the conic section, and, EI being in this case equal to HI, EB shall be equal to EG, or B coincide with G: from which it follows, that if EH be drawn from E perpendicular to either axis, and the angle HEG be made equal to HET on the opposite side of HE, then EG shall meet the conic section in the point G where the circle of curvature and conic section intersect each other.

FIG. 158. n. 1. & 2. 378. COR. IX. To these we may add the following properties of the circle of curvature belonging to any point of a conic section. 4. In the ellipse or hyperbola, let O*a*, the semidiameter parallel to ET the tangent of the section at E, meet EB, any chord of the circle of curvature, in R; bisect EB in *b*; and the rectangle RE*b* shall be equal to the square of O*a*. For, let the diameter through the point of contact meet the circle of curvature again in *e*, join B*e*; and, the angle E*e*B being equal to BET or ERO, the triangles EB*e*, EOR are similar, EB is to E*e* as EO is to ER, and the rectangle REB is equal to the rectangle OE*e*. But E*e* is equal to the parameter of the diameter through E, by the first property of the circle of curvature in the last article: therefore the rectangle OE*e* (or REB) is equal to twice the square of O*a*, and the rectangle RE*b* is equal to the square of O*a*. When EB passes through the focus, ER is equal to half the transverse axis; therefore the chord of the circle of curvature that passes through the focus, the diameter conjugate to that which passes through the point of contact, and the transverse axis of the figure are in continued proportion. It appears likewise, that when the section is an ellipse, if the

circle

circle of curvature at E meet Oa in d, the ſquare of Ed ſhall be equal to twice the ſquare of Oa; therefore Ed is to Oa in the invariable ratio of the diagonal of a ſquare to its ſide, or of the ſquare root of two to unit: and hence, when the point of contact E and the ſemidiameter Oa are given in poſition and magnitude, the center of curvature is readily determined. 5. The right line EG being ſuppoſed to meet the conic ſection in any two points E and G, and the tangents at thoſe points to interſect each other in V, let EB be biſected in b, join Vb; then the angle EVb ſhall be equal to the angle GEO, or to its ſupplement to two right angles; and a circle through E, V and b ſhall always touch the diameter that paſſes through E. For, in the ellipſe and hyperbola, if Op parallel to EG meet the tangent EV in p, the rectangle contained by VE and Ep (or RO) ſhall be equal to the ſquare of Oa, or to the rectangle REb; and EV is to Eb as RE is to RO. Therefore, ſince OR is parallel to EV, the triangles REO, bEV are ſimilar, the angle EVb is equal to REO, and a circle through E, V and b touches EO. In the parabola, let EG be biſected in g; and, ſince Eb is to Eg as the ſquare of EI is to the ſquare of HI, (by art. 373.) or as the ſquare of EV is to the ſquare of Eg; it follows, that the rectangle bEg is equal to the ſquare of EV, and the angle EVb equal to EgV or GEO. Hence when, in any conic ſection, the tangents EV, GV are given, and the diameter through E is given in poſition, the point b and the center of curvature are readily determined. When any two points in a conic ſection, as E and G, and the tangents at theſe points EV and GV, with the circle of curvature belonging to one of them, as E, are given, the ſection is determined by biſecting EB and EG in b and g, joining Vb and Vg, and making the angle bEL equal to bVE ſo that EV and EL may be on oppoſite ſides of Eb: for EL ſhall be a diameter of the ſection, and if it interſect Vg in O, then O ſhall be the center of the figure; but if EL be parallel to Vg, the figure is a parabola. When b and g are on different ſides of E, the figure is an hyperbola; and when theſe points are on the ſame ſide of E, it is an hyperbola or ellipſe according as the angle EVb is greater or leſs than EgV. When two points E and G of a conic ſection are given, with

Fig. 158. n. 3.

Fig. 158. n. 1. 2. & 3.

with EV the tangent at E, and the points B and D where the circles of curvature at E and G meet EG, the tangent GV is determined, by drawing from G to the tangent EV a right line GV so that GV may be to EV in the ſubduplicate ratio of GD
Fig. 155. to EB. For if HI the tangent at H meet GV the tangent at G in *i*, H*i* and HI ſhall be equal; and if the circle of curvature at G meet EG in D, GD ſhall be to EB as the ſquare of G*i* is to the ſquare of EI, (by art. 373.) or as the ſquare of GV
Fig. 159. is to the ſquare of EV. 6. Let EB be the chord of the circle
n. 1. & 2. of curvature that paſſes through S the focus of the conic ſection; let BX parallel to the tangent ET meet EX perpendicular to ET in X; and if XZ be perpendicular to EB in Z, then ſhall EZ be equal to the parameter of the tranſverſe axis of the figure. For let EK be equal to the half of this parameter; then, ſince EB is to EZ as the ſquare of EB to the ſquare of EX, or as the ſquare of ER (which in this caſe is equal to half the tranſverſe axis of the figure) to the ſquare of EN, and, conſequently, as the ſquare of O*a* to the ſquare of half the conjugate axis, or (by the fourth property of the circle of curvature) as the rectangle *b*ER is to the rectangle KER, that is, as *b*E is to KE; and, ſince EB is equal to 2*b*E, it follows, that
Fig. 158. EZ is equal to 2EK. In the parabola, let SP perpendicular on
n. 3. the tangent at E meet it in P, and A be the vertex of the figure; then EB ſhall be to EZ as the ſquare of SE to the ſquare of SP, or as SE is to SA; and ſince EB is equal to 4SE, (art.
Fig. 159. 371.) EZ is equal to 4SA. Hence, when the focus S, point
n. 1. & 2. of contact E, the tangent ET, and EB the chord of the circle of curvature through S are given, the principal parameter of the figure is readily determined; or, when the reſt are given, the center of curvature is eaſily found. 7. Let S and *ſ* be the two *foci* when the ſection is an ellipſe or hyperbola; let ON parallel to the tangent ET from the center O meet EN perpendicular to ET in N; let C be the center of the circle of curvature at E; join SC and *ſ*N: then ſhall the angle ESC be equal to EN*ſ*. For the rectangle CEN is equal to the ſquare of O*a*, (by the fourth property,) the ſquare of O*a* is equal to the rectangle contained by SE and *ſ*E; therefore CE is to SE as *ſ*E is to EN; and, ſince the angle SEC is equal to *ſ*EN, the triangles

angles SEC, NE*f* are ſimilar, and the angle ESC is equal to NE*f*. Hence the center of curvature at E is readily found when the two *foci* are given. 8. If EQ be taken upon EB, that chord of the circle of curvature which paſſes through the focus S, equal to one fourth part of EB; then SQ, SE and the tranſverſe axis of the ſection ſhall be in continued proportion. For, ſince the rectangle R*Eb* is equal to the rectangle contained by SE and *f*E, SE is to EQ (or one half of *Eb*) as 2ER is to *f*E; from which it follows, that SQ is to SE as SE is to 2ER, which is equal to the tranſverſe axis of the figure. Hence, when the focus S, the point of contact E and tangent ET with EB that chord of the circle of curvature which paſſes the focus are given, let EQ be one fourth part of EB, draw SP perpendicular to ET in P, let P*r* biſect SE in *r*, let PO be taken upon P*r* (the ſame or the coutrary way from P with P*r* according as EQ is leſs or greater than ES) equal to one half of the third proportional to SQ and SE; and O ſhall be the center of the figure. According as EQ is greater or leſs than ES the figure is an hyperbola or ellipſe; and, when EQ is equal to ES, it is a parabola, the vertex of which is determined by making the angle PSA equal to PSE and PA perpendicular to SA in A. FIG. 158. n. 3.

376. COR. X. The variation of curvature at any point of a conic ſection, is always as the tangent of the angle contained by the diameter that paſſes through the point of contact, and the perpendicular to the curve at the ſame point. Thus the variation of curvature at E is as the tangent of the angle GEO, EG being a perpendicular to the curve, and EO a diameter. For if the tangent at G interſect the tangent ET in V, and *b* be the center of curvature, the angle EV*b* ſhall be equal to GEO, or to its ſupplement to two right angles, by the fifth property of the circle of curvature demonſtrated in the laſt article. The variation of curvature at E is as the tangent of the angle EVB, or of EV*b*, (becauſe the tangent of the latter angle is always one half of the tangent of the former;) and, conſequently, as the tangent of the angle GEO. Hence the variation of curvature vaniſhes at the extremities of either axis, and is greateſt when the acute angle contained by the diameter OE and the tangent ET is leaſt. When the ſection is a parabola and S is the focus, FIG. 160. FIG. 161.

focus, the angle GEO is equal to GES; and the variation is as the variation of the angle contained by the right line drawn from the point of contact to the focus and the perpendicular to the curve. Hence, when the point of contact E, the tangent EV, with the curvature at E and its variation are given, the parabola is determined: For, BE and EV being given, biſect EB in *b*, join V*b*, make the angle BEb equal to *b*VE, and let Eb meet the circle of curvatnre in b; take ES upon Eb equal to one fourth part of Eb: and S ſhall be the focus of the parabola. If EB meet the axis of the parabola in N, and EL be perpendicular to the axis in L, the variation of curvature ſhall be as the tangent of the angle ENL, or as the ordinate EL directly and the parameter of the axis inverſely. The meaſure of the angle of contact contained by the parabola and circle of curvature at any point E, is as EL the ordinate to the axis directly and the cube of SE the diſtance of E from the focus inverſely.

Fig. 162. 377. Cor. XI. When EB does not meet with the curve FK, but is its aſymptote, any circle being deſcribed touching ET in E, a greater circle ſhall always paſs between it and the curve EM; and the greater this circle is, the cloſer ſhall its contact be with the curve EM. For, ſince the curve FK produced paſſes without any circle EQB how great ſoever that can be deſcribed through E, EM muſt always paſs betwixt ER and the tangent ET. This is the caſe in which the curvature is ſaid to be infinitely little (being leſs than that of any circle) or the ray of curvature infinitely great. The point E in the curve HME is a point of contrary flexure when the two hyperbolic branches of the curve FK proceed along the aſymptote EB on different ſides of it and with oppoſite directions, as in the common hyperbola. But when theſe branches are on the ſame ſide of the aſymptote and proceed along it with oppoſite directions, E is a cuſpid of the firſt kind; when they proceed with the ſame direction on different ſides of the aſymptote, the curve EM has its concavity turned the ſame way on both ſides of E; when thoſe branches proceed along the aſymptote on the ſame ſide of it and with the ſame direction, E is a cuſpid of the ſecond kind: and in all theſe caſes the curvature at E is leſs than that of any circle. Of the firſt caſe we have an example in the vertex

Fig. 153.

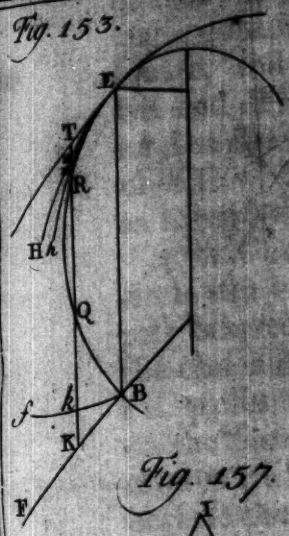


Fig. 154.

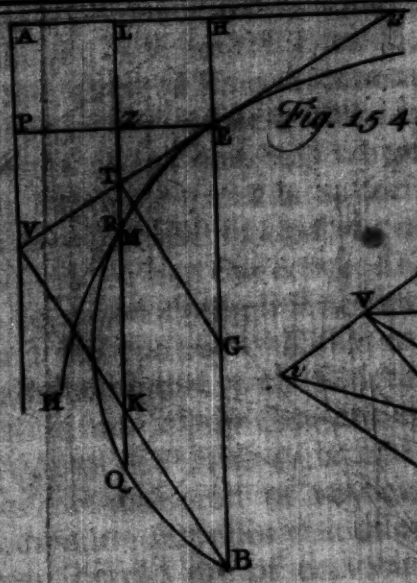


Fig. 155.

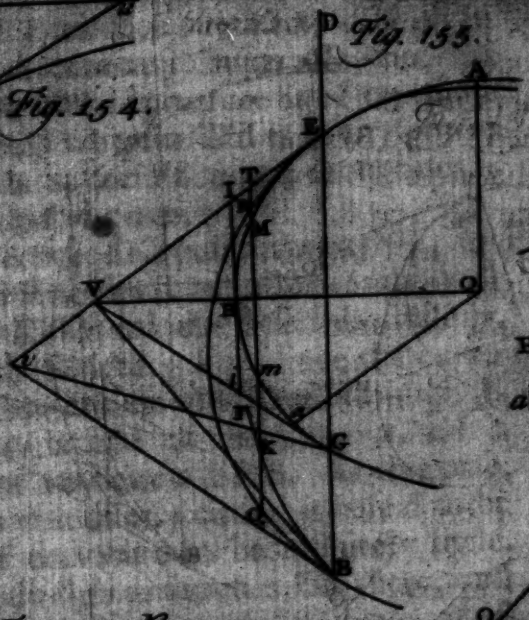


Fig. 156.

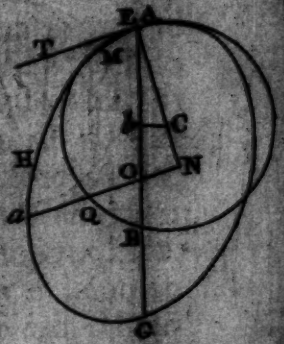


Fig. 157.

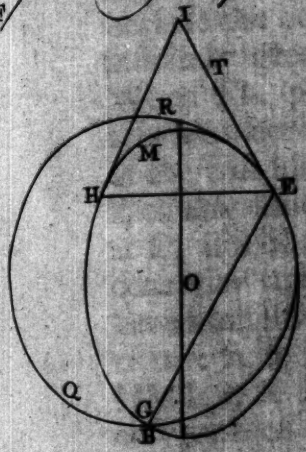


Fig. 158. N.1

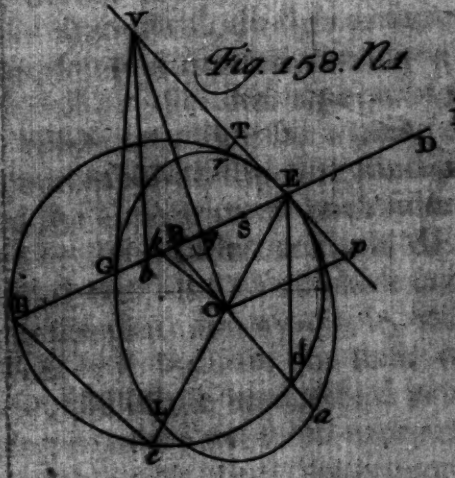


Fig. 158 N.2

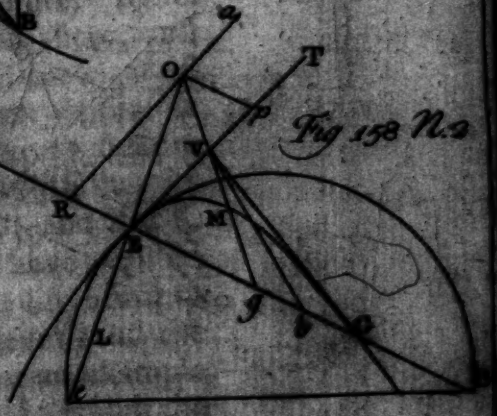


Fig. 158. N.3

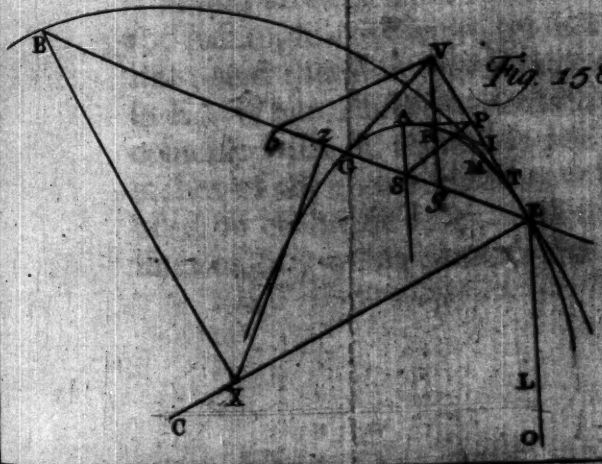
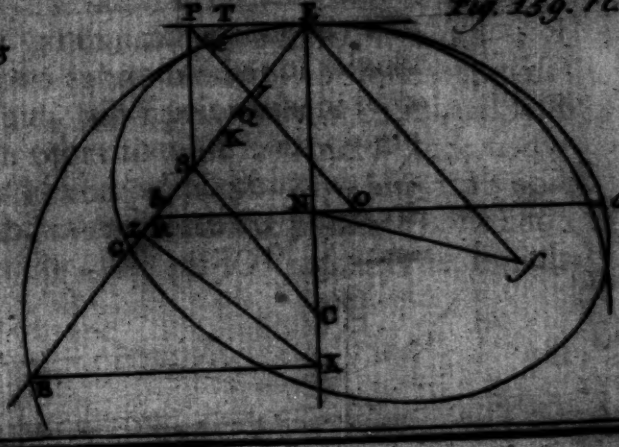
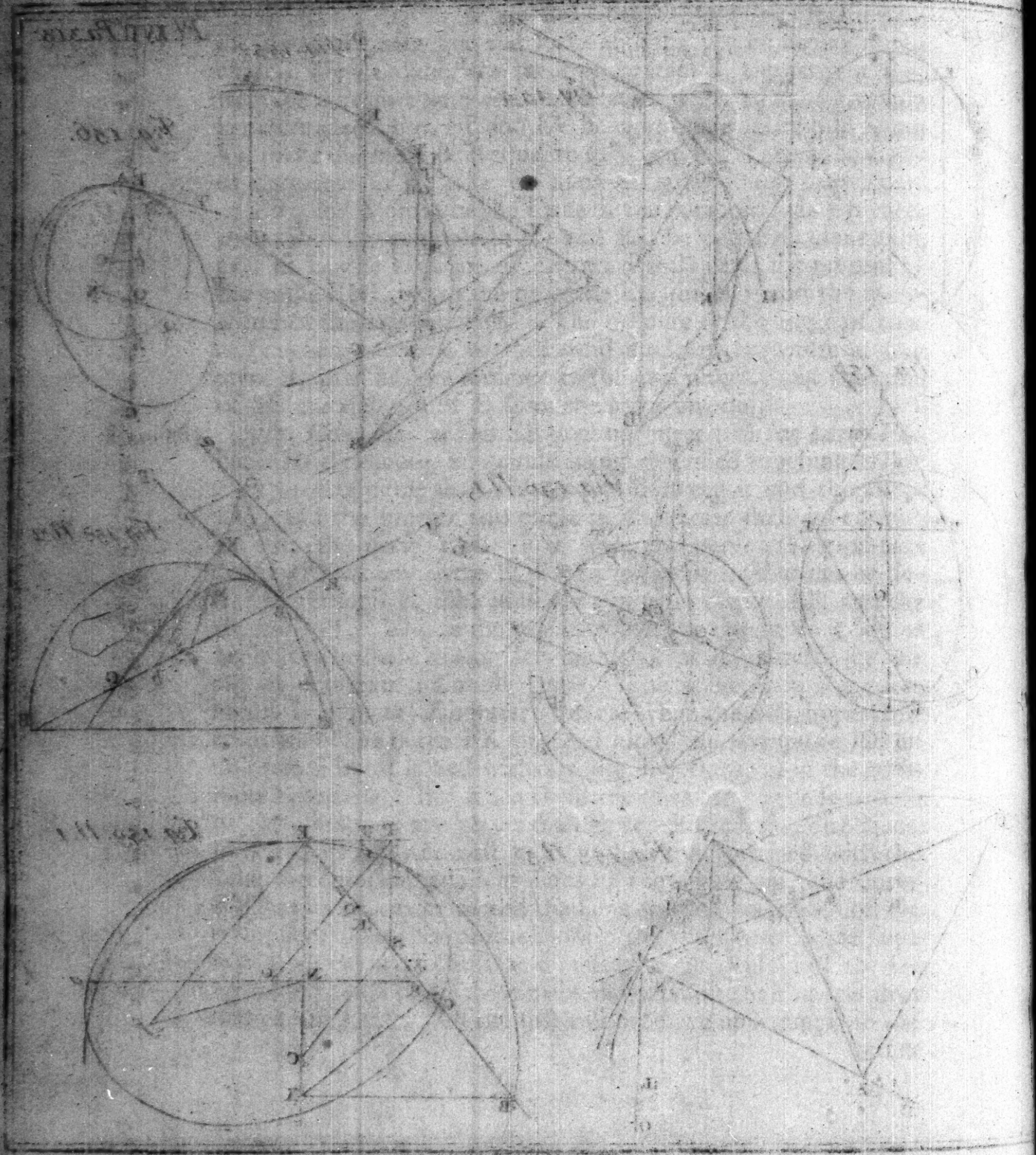


Fig. 159. N.1





vertex of the cubic parabola, wherein the cube of ET being equal to the ſolid contained by TM and a given ſquare, the rectangle contained by TK and ET muſt be equal to this given ſquare, and the curve FK is a common hyperbola that has EB and ET for its aſymptotes. The curvature is of the ſame kind at the vertex of any parabola wherein TM is as any power of ET whoſe exponent exceeds 2, for FK in all thoſe caſes is an hyperbola of which EB is an aſymptote.

378. Cor. XII. When the curve FK paſſes through E, no circle can be deſcribed through E ſo ſmall but a leſs circle ſhall paſs between it and the curve EM; and the leſs this circle is, the cloſer ſhall its contact with EM be. For, ſince the curve FK paſſes within any circle that can be deſcribed through E on the ſame ſide of ET, the arch EM of the curve EMH is always within ER the arch of any circle ERB. In this caſe, becauſe the curvature ſurpaſſes that of any circle, it is ſaid to be infinitely great, or the ray of curvature to be infinitely little. When FK paſſes through E, it always touches EB there, and if it has a continued curvature at E, the curve HME has a cuſpid of the firſt kind at E; if FK has a point of contrary flexure at E of any kind, HME has likewiſe a point of contrary flexure at E; if FK has a cuſpid of the firſt kind at E, HME has its concavity turned the ſame way on both ſides of E; and if FK have a cuſpid of the ſecond kind at E, HME ſhall have a cuſpid of the ſame kind there; and in all thoſe caſes the curvature of HME at E ſurpaſſes that of any circle, of whatever kind the curvature of FKE at E be. Of the firſt, we have an example at the vertex or cuſpid of the ſemicubical parabola, in which the cube of ET being equal to the ſolid contained by the ſquare of TM and an invariable right line, the ſquare of TK is equal to the rectangle contained by ET and the ſame invariable right line; and therefore FK is a common parabola that touches EB in E. When FKB interſects EB in two points neither of which coincide with E, then E is not a cuſpid, but a point where two arches of the curve HME touch each other that are both continued on each ſide of E. Of this kind there are no points in the lines of any order beneath the fourth, as there are no points of con- FIG. 162.

contrary flexure, or cuſpids, in the lines of any order beneath the third.

Fig. 163. 379. Cor. XIII. Suppoſe now that EMH is any line of the third order; let ET the tangent at E meet the curve in A, EG any right line through E meet it in the points E, G and *g*, and TM parallel to EG meet it in M, *m* and m: then, if EB be taken upon EG on the concave ſide of the arch EM, ſo that the rectangle AEB be to the rectangle GE*g* as the ſolid contained by AT and the ſquare of ET is to the ſolid contained by TM, T*m* and Tm, a circle deſcribed upon the chord EB ſo as to touch ET ſhall be the circle of curvature at E. For theſe ſolids are in an invariable ratio to each other when ET and EG are given in poſition and TM is alwas parallel to EG, by the properties of the lines of the third order; and, ſince the rectangle MTK is ſuppoſed equal to the ſquare of ET, it follows, that the rectangle ATK is to the rectangle *m*Tm in the ſame invariable ratio: Therefore the curve FKB ſhall meet EG in B ſo that the rectangle AEB ſhall be to GE*g* in the ſame ratio. In all the lines of this order the curvature at the confine of contrary flexure is leſs than the curvature of any circle, or

Fig. 164. n. 1. is of that kind which was deſcribed in cor. 11. For when E is at this confine, the tangent ET meets the curve in the point E only; the cube of ET is to the ſolid contained by TM, T*m* and Tm in an invariable ratio: therefore the rectangle ETK is to *m*Tm in the ſame ratio; conſequently EG is an aſymptote of the curve FK in this caſe, and (by cor. 11.) the

Fig. 164. n. 2. curvature at E is leſs than that of any circle. It is the contrary when E is a cuſpid in any line of this order: for the curvature at the cuſpid is always greater than that of any circle, or is of the kind deſcribed in the laſt corollary; becauſe, in this caſe, while T approaches to the cuſpid E, the ratio of ET to T*m* may become greater than any given ratio; conſequently, the ratio of Tm to TK may likewiſe become greater than any given ratio, and when m comes to *g*, K muſt come to E: therefore the curve FK paſſes through E, and the curvature at the cuſpid E is greater than that of any circle, by the laſt corolla-

Fig. 163. ry. In theſe lines the variation of curvature is found by determining the tangent of the curve FKB at B when GET is a right angle;

angle; and it is eaſy to ſhew, that if GR, *gr* touch the curve at G and *g*, and meet EA in R, and *r*, reſpectively; and if BV the tangent of FKB meet EA in V; then EV ſhall be to EA as the rectangle R*E*r is to the ſum, or difference, of the rectangle R*A*r and the ſquare of E*A*.

380. Cor. XIV. When EMH is a geometrical curve, BKF is likewiſe geometrical, ſince TM, TE and TK are in continued proportion; and ſince the point B is determined by the interſection of a geometrical curve with the right line EB, it follows, that, when the angle BET is given, the curve in which B is always found is likewiſe geometrical; and according as EMH is geometrical, or mechanical, the curve in which the center of curvature of EMH is always found is geometrical, or mechanical. By this propoſition and the preceeding corollaries, the curvature of EMH and its variation, with the degree of contact of the curve and circle of curvature, may be determined when EMH is geometrical. We now proceed to the theorems by which theſe are determined with equal facility when it is mechanical. Fig. 149. & 150.

381. Let E*m* be a common parabola, ET the tangent at E, E*d* the diameter through E, EB the parameter of this diameter, and let any right line T*m* parallel to E*d* meet the tangent in T and the parabola in *m*. Let *ef* a right line given in poſition meet E*d* in *e* and T*m* in *f*; then, if *fn* be taken upon *fm* ſo that the triangle *efn* be always equal to the rectangle contained by T*m* and a given right line DG, the point *n* ſhall be always found in a right line *en* given in poſition. For, ſince the rectangle contained by T*m* and EB is always equal to the ſquare of ET, (by the known property of the parabola,) the triangle *efn* is to the ſquare of ET as DG is to EB. Becauſe ET and *ef* are given in poſition, and *f*T is always parallel to *e*E, the ſquare of ET is to the ſquare of *ef* always in a given ratio; conſequently, the triangle *efn* is to the ſquare of *ef*, and *fn* is to *ef*, in a given ratio: Therefore *en* is a right line given in poſition. And, converſely, when ET, *ef* and *en* are given in poſition, if the rectangle contained by T*m* and a given right line DG be always equal to the triangle *efn*, the point *m* ſhall be in a parabola that touches ET in E ſo as to have its diameters Fig. 165.

parallel to T*u*, and the parameter of the diameter through E in the ſame ratio to DG as the ſquare of ET is to the triangle *eſn*; for ſuppoſing EB to be to DG in this ratio, the rectangle contained by T*m* and EB ſhall be equal to the ſquare of ET.

382. Let the baſe AP flow uniformly, and its fluxion be repreſented by any given right line DG; let PN the ordinate of the figure DeNP always meaſure the fluxion of PM the ordinate of DEMP as in prop. 20. let *en* the tangent of *e*N at E and *ef* parallel to DP meet PN in *n* and *f*, and ET the tangent of EM meet it in T: then, when DP becomes equal to DG, *fn* ſhall meaſure the fluxion of the ordinate D*e*, (by prop. 14.) or the ſecond fluxion of DE; and ET ſhall meaſure the fluxion of the curve HE.

PROP. XXXIII.

The baſe AD *being ſuppoſed to flow uniformly, let the ſecond fluxion of the ordinate* DE *be to the fluxion of the curve* HE *as the fluxion of the curve is to* E*b*; *and if* EB *be taken upon* DE *from* E *on the concave ſide of the curve equal to* 2E*b*, *the circle of curvature at* E *ſhall paſs through* B.

Fig. 165.

Let the rectangle contained by T*m* and the given right line DG be always equal to the triangle *eſn*, and, by the laſt article, the point *m* ſhall be always found in a parabola E*m* that touches ET in E, whoſe diameter through E is E*d*; and the parameter of this diameter muſt be equal to EB, becauſe it is to DG as the ſquare of ET (or the rectangle contained by *ſn* and E*b* when DP becomes equal to DG) is to the triangle *eſn* (or the rectangle contained by *ſn* and one half of *ef*, or DG;) that is, as EB is to DG. But no parabola can be deſcribed through E betwixt the curve EM and the parabola E*m*. For, let E*u* be any other parabola deſcribed through E, touching ET, and having E*d* for its diameter; let TM meet this parabola in *u*, and the rectangle contained by T*u* and DG be always equal to the triangle *eſx*, then the point *x* ſhall be in a right line *ex* given

given in position, by the last article; if Eb be the parameter of the diameter of *Eu* that passes through E, the triangle *esx* shall be to the square of ET as DG is to Eb, and (because the square of ET is to the triangle *esn* as EB is to DG) the triangle *esx* to *esn* as EB is to Eb, so that *sx* is to *sn* in the same ratio. The rectangle contained by TM and DG is always equal to the trilineal area *esN*, by what was shewn in prop. 20. Therefore the right lines TM, T*m* and T*u* are in the same proportion to each other as the area *esN*, the triangle *esu*, and the triangle *esx*. But, because *eu* is the tangent of the curve *eN*, if we suppose PM to move towards DE, the area *esN* and triangle *esn* shall either become both greater or both less than the triangle *esx* (by art. 181.) according as *sn* is greater or less than *sx*, or Eb is greater or less than EB; consequently, TM and T*m* shall either become both greater or both less than T*u* while M approaches to E; and therefore the parabola *Eu* cannot pass between the curve EM and parabola *Em*. In the same manner, no other parabola can pass between EM and *Em*; but an indefinite number of parabolas can be described between EM and any other parabola *Eu*, as an indefinite number of right lines may be drawn through *e* betwixt *en* and *ex* within the angle *nex*. Therefore the curve EM and the parabola *Em* have the same circle of curvature, by the latter part of art. 373. and this circle is that which is described upon EB so as to touch the right line ET, by the same article.

383. Cor. I. As all curves that pass through E have the same tangent when the first fluxions of their ordinates are equal, the fluxion of the base being given; so they have the same circle of curvature at E when the second fluxions of their ordinates are likewise equal. Thus, the motion with which the base flows being given, the position of the tangent depends on the motion with which the ordinate flows, and the flexure of the circle of curvature depends on the acceleration or retardation of this motion. When the angle BET and the fluxion of the base are given, the ray of curvature EC is reciprocally as the second fluxion of the ordinate; because EC is to *Eb* (which is reciprocally as *sn*) as ET is to DG. When the fluxion of the base is constant, the solid contained by EC the ray

of curvature, DG which measures the fluxion of the base, and *ſſ* that measures the second fluxion of the ordinate, is always equal to the cube of ET which measures the fluxion of the curve.

FIG. 167. If the curve ER be generated from *er* in the same manner as EM is generated from *e*N, (that is, if the rectangle contained by DG and TR be always equal to the area *eſf*,) the contact of EM and ER shall be always one degree closer than that of *e*N and *er*. Thus, if *e*N and *er* pass through the same point *e*, EM and ER shall have the same tangent; when they have the same tangent, EM and ER have the same curvature; when *e*N and *er* have the same curvature, the contact of EM and ER with their common circle of curvature is of the same kind, or they have the same variation of curvature, and so on.

FIG. 166. 384. COR. II. The second fluxion of the curve HE is to the first fluxion of the curve as the fluxion of the ordinate DE is to *Eb*: And, S being any given point in the line D*d*, if SK be always perpendicular to the tangent of HE in K, the fluxion of the perpendicular SK shall be to the fluxion of SE as SK is to *Eb*, or as SE is to the ray of curvature EC. For, let any ordinate *pm* meet the curve EM in m and *e*N in n; let *nſ* parallel to the base meet GM in *ſ*, and S*k* be perpendicular to the tangent at m in *k*: then D*ſ* and D*ſ* shall be parallel to the tangents of the curve HEm at E and m, respectively, and shall measure the fluxions of the arches HE and Hm, the fluxion of the base being measured by DG, or *eſ*. Let *ny* be perpendicular to D*ſ* in *y*; and, since the fluxion of D*e* is measured by *ſn*, the fluxion of D*ſ* (or the second fluxion of the curve HE) shall be measured by *ſy*, by art. 193. But *ſy* is to D*e* as *ſn* is to D*ſ*, or (by this proposition) as D*ſ* is to *Eb*. Therefore the second fluxion of the curve (which is measured by *ſy*) is to the fluxion of the curve (or D*ſ*) as the fluxion of the ordinate (or D*e*) is to *Eb*. This likewise appears from art. 96. because the fluxion of the square of ET, or D*ſ*, is equal to the fluxion of the square of D*e*, DG being invariable. Let the angle SK*x* be made equal to SET or D*ſ*G, and K*x* shall be the tangent of the curve K*k* that passes always through the intersections of the tangents of HEM and the perpendiculars from S, by art. 211. The angle KS*k* is equal to *ſ*D*ſ*, and the angular velocity of SK about S is equal to

the

Pl. XIII. Pa. 394.

Fig. 159. N. 2

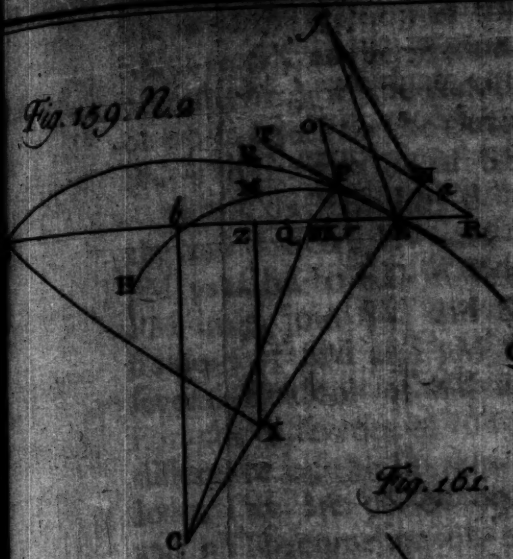


Fig. 160.

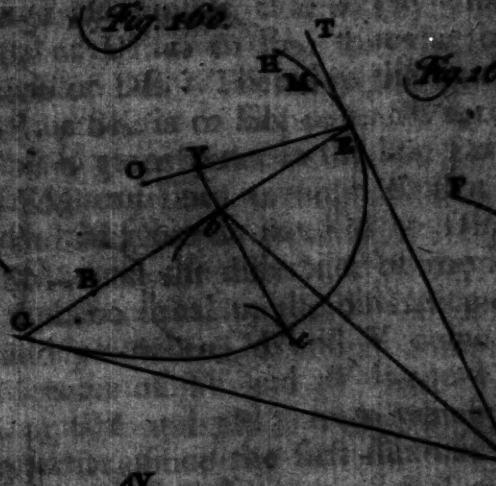


Fig. 162. N. 1

Fig. 161.

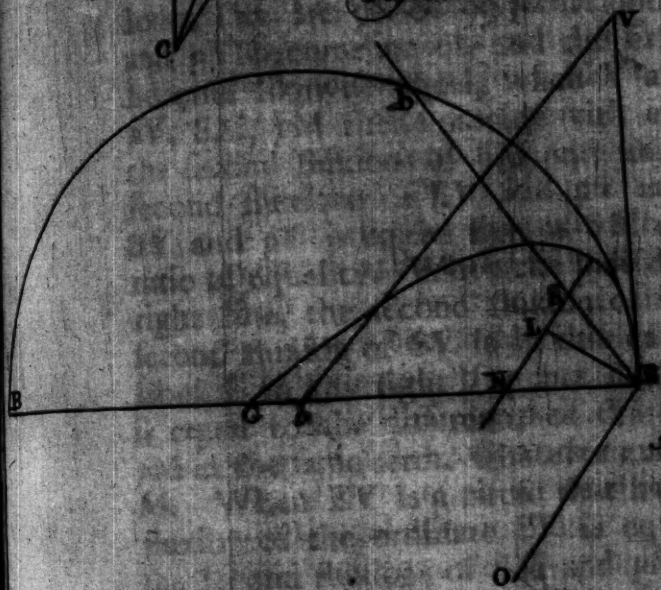


Fig. 163.

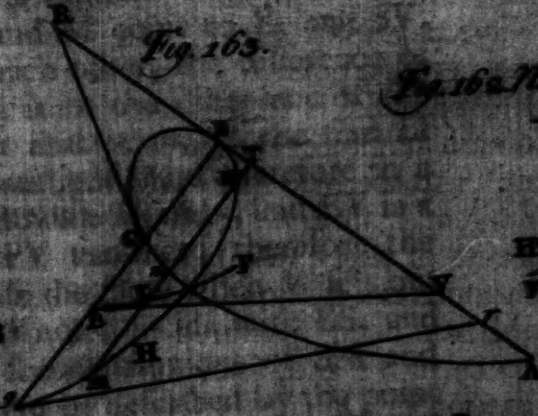


Fig. 162. N. 2

Fig. 165.

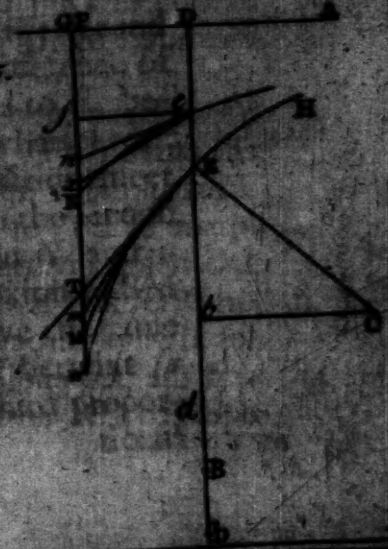


Fig. 164. N. 1

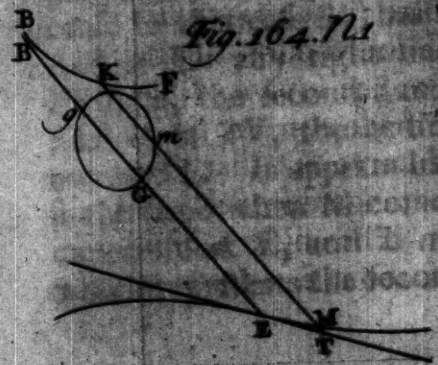
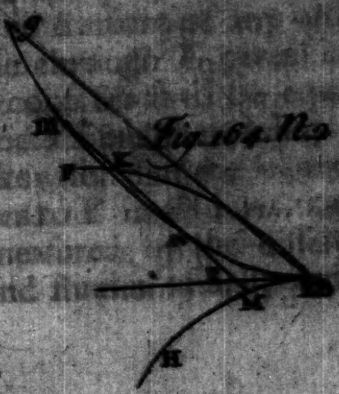
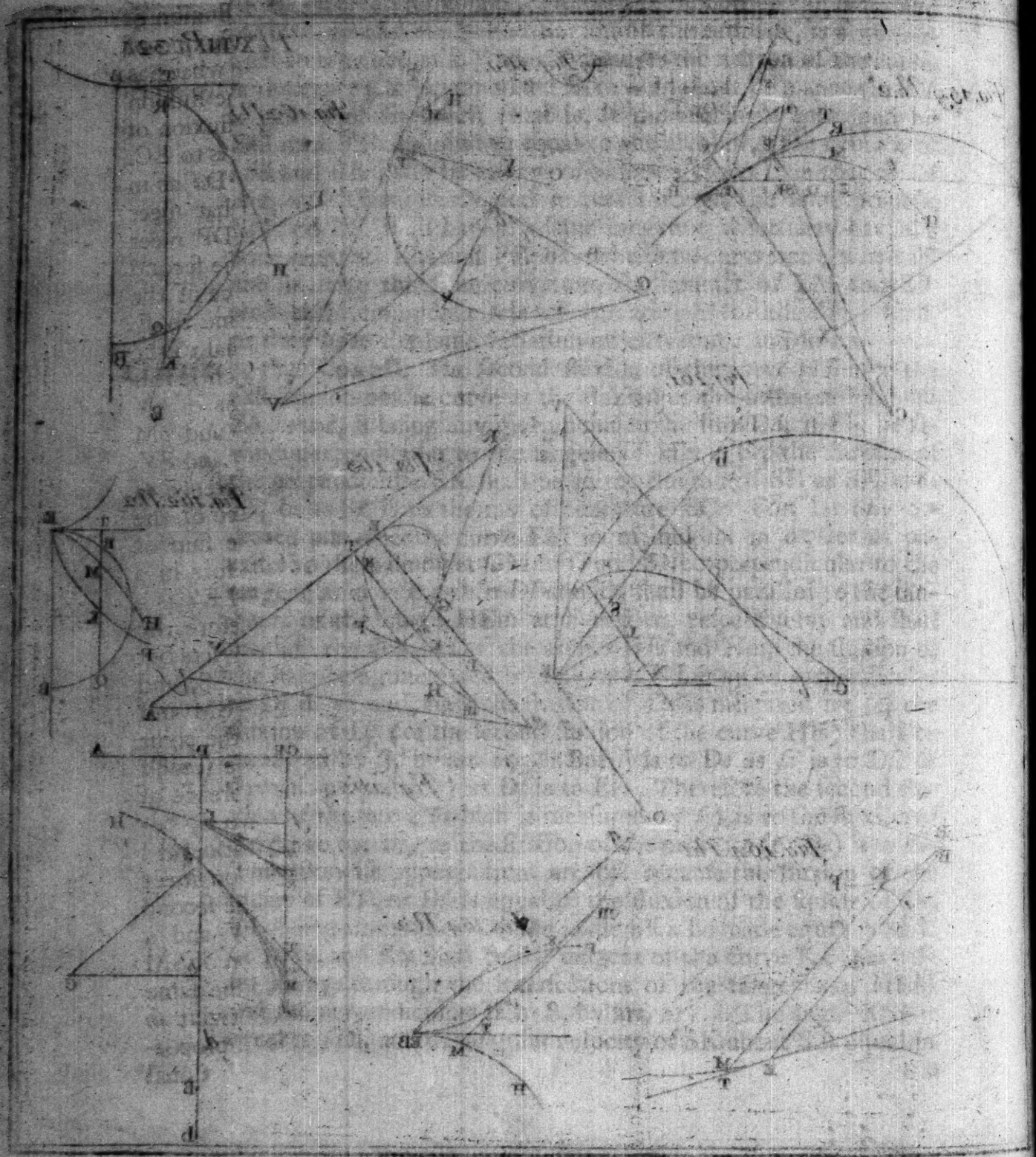


Fig. 164. N. 2





the angular velocity of D*ſ* about D. Therefore the fluxion of SK is to the fluxion of D*ſ* as SK is to D*ſ*; and the fluxion of SK is to SK as the fluxion of D*ſ* is to D*ſ*, that is, (by what has been ſhewn,) as the fluxion of DE is to E*b*. But the fluxion of SE is equal to the fluxion of DE. Therefore the fluxion of SK is to the fluxion of SE as SK is to E*b*, or as SE is to EC.

385. Cor. III. Let S be a given point in the line D*d* as in the laſt article, EV and EM any lines through E that meet *p*M parallel to SD in V and M; let S*p* parallel to DP meet PM in *p*, join SV and SM; and the difference of the ſecond fluxions of SM and *p*M ſhall be equal to the difference of the ſecond fluxions of SV and *p*V when M and V come to E. For, the difference of the ſquares of SV and *p*V is equal to the difference of the ſquares of SM and *p*M: from which it follows, (by art. 96. & 99.) that, ſince the firſt fluxions of SV and *p*V become equal, and the firſt fluxions of SM and *p*M become likewiſe equal, when V and M come to E, and SV, *p*V, SM, *p*M then coincide with each other, the difference of the ſecond fluxions of SM and *p*M is to the difference of the ſecond fluxions of SV and *p*V at that term, as the ſum of SV and *p*V is to the ſum of SM and *p*M, and therefore in a ratio of equality. When the baſe flows uniformly, and EV is a right line, the ſecond fluxion of PV vaniſhes; therefore the ſecond fluxion of SV is in this caſe the ſame when V ſets out from E in any right line that does not coincide with ES, and is equal to the difference of the ſecond fluxions of SM and *p*M at the ſame term, whatever curve be deſcribed by the point M. When EV is a circle that has its center in S, the ſecond fluxion of the ordinate PV is equal to the ſame difference of the ſecond fluxions of SM and *p*M, when M comes to E, becauſe in this caſe SV has no fluxions of any order whatſoever: And if E*u* any right line through E given in poſition meet PV in *u*, the ſecond fluxionof S*u* ſhall be equal to the ſecond fluxion of PV, the ordinate from the circle, when V and *u* come to E. It appears likewiſe, that the ſecond fluxion of SM is the ſame when M comes to E in all lines that have the ſame curvature at E, and is meaſured by the difference betwixt *ſn* which repreſents the ſecond fluxion of DE, and a third proportional

tional to SE and DG (which represents the invariable fluxion of AD;) because the fluxion of the circular arch EV becomes equal to the fluxion of AD when V comes to E, and the second fluxion of PV the ordinate of the circle is to DG as DG is to SE the radius of the circle, by this proposition.

FIG. 167. 386. COR. IV. The rest remaining as in art. 384. let EB be perpendicular to the tangent ET and to AD, and, since *fy* which measures the fluxion of ET (or the second fluxion of the curve HE) vanishes, the fluxion of the rectangle contained by *fn* and EC (which in this case coincides with E*b*) shall likewise vanish, because that rectangle is equal to the square of ET; and the fluxion of EC the ray of curvature is to the fluxion of *fn* as EC is to *fn*, by prop. 3. Let *er* be a parabola that has the same curvature with *e*N at *e*, and the rectangle contained by TR and DG be always equal to the area *efr*, as in art. 383. Let the rectangles MT*k*, *m*TQ, R*Tk* be each equal to the square of ET, as in art. 366. and let T*k* meet B*t* the tangent of B*k* (or of BK) in *t*. Then, the fluxion of the base being constant, the fluxion of *fn* shall be represented by 2*nr*, or 6*m*R, by art. 255. But, since T*m* is to TR as T*k* is to TQ, 6*m*R (which measures the fluxion of *fn*) is to 6*k*Q as T*m* is to T*k*; and therefore, as the rectangle *m*TQ is to the rectangle contained by T*k* and TQ, or EB. The rectangle *m*TQ is equal to the square of ET, or (by this proposition) to the rectangle contained by *fn* and E*b*; consequently, the fluxion of *fn* is to 6*k*Q as *fn* is to 2T*k*, and the fluxion of the ray of curvature EC is to 6*k*Q as EC is to 2T*k*, or as EB is to 4T*k*: from which it follows, that the fluxion of the ray of curvature is measured by three halves of Q*t*, the fluxion of the curve being measured by ET or BQ; and that the variation of curvature, according to Sir ISAAC NEWTON, (who measures it by the ratio of the fluxion of the ray of curvature to the fluxion of the curve,) is measured by three halves of the tangent of the angle QB*t* in which the curve BK intersects the circle of curvature, as was observed in art. 369. But, since the curvature itself is reciprocally as the ray of the circle of curvature, if we should therefore measure its variation by the ratio of its fluxion to the fluxion of the curve, this variation would be as the tangent of the angle QB*t*

direct-

directly and the ſquare of the ray of curvature inverſely, and would be as the meaſure of the angle of contact contained by the curve and circle of curvature; becauſe, when a quantity A is always inverſely as another quantity B, its fluxion is as the fluxion of B directly and the ſquare of B inverſely. However, to avoid confuſion, we have conformed to Sir ISAAC NEWTON's explication of the variation of curvature.

387. COR. V. When *en* the tangent of *eN* is parallel to the baſe, and the rectangles contained by DG and TM, and by DG and T*m*, are reſpectively equal to the areas DPN*e*, DP*ne*, as in prop. 20. then E*m* becomes a right line, and the curvature of EM is of the kind that is leſs than the curvature of any circle, which was deſcribed in art. 377. In this caſe, the ſecond fluxion of the ordinate DE vaniſhes; and (provided the curve be continued on both ſides of DE) if the number of the fluxions of DE of ſucceſſive orders that vaniſh be an odd number, then E is a point of contrary flexure; but if it is an even number, E is not a point of that kind, as was ſhewn in art. 266. FIG. 165.

388. COR. VI. The ſame things being ſuppoſed as in art. 384. it follows from art. 202. & 208. that the velocity of the angular motion of the tangent of HEm (or of SK about S) when m ſets out from E, is the ſame when the point n deſcribes any curve that has the ſame tangent at *e*, or when m deſcribes any line that has the ſame tangent and curvature at E, the fluxion of AD being given. Therefore the angular velocity of the tangent of EM at E is equal to the angular velocity of the tangent of the circle of curvature, or of the ray of curvature about the center C, the fluxion of AD being given, becauſe the angular velocity of the tangent of a circle is equal to the angular velocity of the ray drawn from the center to the point of contact, by art. 18. FIG. 166.

In like manner, if B be any point in the circle of curvature, and the right lines EM, BM revolve about E and B ſo that their interſection M deſcribe the curve HME, the angular velocities of EM and BM ſhall be equal when M comes to E. For, ſuppoſe the arch ER of the circle of curvature to be within EM the arch of the curve EMH; and, if the angular velocity of EM about E be ſaid to be leſs than the angular velocity of BM about B at the term when M comes to E, FIG. 168.

E, let it be leſs in the ratio of EB to EZ, and (by art. 209.) it ſhall be equal to the angular velocity of ZM about Z at that term, and therefore leſs than the velocity of *b*M about *b* at the ſame term, *b* being any point betwixt B and Z. Upon E*b* deſcribe a circle E*rb* that touches ET, produce EM till it meet this circle in *r*, join *br* and *b*M; and, ſince the arch EM is within this circle E*rb*, (art. 364.) the angle MET, or E*br*, is always greater than E*b*M while M deſcribes EM: therefore the angular velocity of EM about E cannot be leſs than the angular velocity of *b*M about *b* when M comes to E; but it was ſuppoſed equal to the angular velocity of ZM about Z at that term; and theſe are contradictory. In the ſame manner, it is ſhewn, that the angular velocity of EM about E is not greater than the angular velocity of BM about B when M comes to E, and that theſe velocities are equal likewiſe when EM is within the circle of curvature ER.

389. The various properties of the circle ſuggeſt various methods of determining the circle of curvature. The two following propoſitions are deduced from the properties of this circle that were deſcribed in the laſt article, and ſometimes give more ſimple conſtructions for determining the ray of curvature than the preceeding propoſitions.

PROP. XXXIV.

Let S *be any given point in the plane of the curve* BL, *and* SP *be always perpendicular from it on* LP *the tangent of* BL; *let* SV *perpendicular to* SL *meet* LC *the ray of curvature in* V: *and* LC *ſhall be to* LV *as the angular velocity of* SL *about* S *is to the angular velocity of* SP.

FIG. 169.

Let CI perpendicular to SL meet it in I, and, ſince the angle LCI is equal to SLP, it follows from what was ſhewn in prop. 18. that when L deſcribes the curve BL, the angular velocity of CL about C is equal to the angular velocity of IL about I, which is to the angular velocity of SL about S as SL is to IL,

IL, (by art. 209.) or as LV to LC. But the angular velocity of CL about C is equal to the angular velocity of SP about S, by art. 388. Therefore LC is to LV as the angular velocity of SL about S is to the angular velocity of SP about S, or, SA being a right line given in position, as the fluxion of the angle ASL is to the fluxion of the angle ASP. This may be demonstrated likewise from art. 384. where it was shewn, that LI is to SP as the fluxion of SL is to the fluxion of SP; from which it follows, that LI is to LS (or LC to LV) in the ratio compounded of that of the fluxion of SL to the fluxion of SP and that of SP to SL; that is, (by art. 202. 208. 211. & 212.) as the fluxion of the angle ASL to the fluxion of ASP.

390. Cor. I. Let LI be bisected in Q, and PM be the ray of the circle of the same curvature with DP (the line in which P is always found) at P; then shall SQ, SL and 2PM be in continued proportion. For, let PT be the tangent of DP at P and ST be always perpendicular to PT, let PM meet LC in K; then, because the angles SPT, SLP are equal, (art. 208. & 209.) and KPT, KLP are right, the angles KPS, KLS are equal, a circle passes through the four points K, L, P, S, and the angle KSP is right. Therefore, by this proposition, MP is to KP, or SL, as the angular velocity of SP is to the angular velocity of ST, or as the fluxion of the angle ASP to the fluxion of 2ASP — ASL, because the angles PST, LSP are always equal. But SL is to LI (or 2LQ) as the fluxion of ASP is to the fluxion of ASL. Therefore PM is to SL as SL is to 2SQ, or SQ, SL and 2PM are in continued proportion.

391. Cor. II. When the angular velocity of SL is to the angular velocity of SP in any invariable ratio, LC is to LV, or LI to LS, in the same invariable ratio. Thus, when AL is a common parabola, and A is the vertex, S the focus, the angle ASL is double of ASP and LI is double of LS; which agrees with what was shewn in art. 371. When AL is the logarithmic spiral, and S is the center of the spiral, the angular velocity of SP is equal to the angular velocity of SL, because the angle LSP is invariable; therefore, in this figure, C coincides with V, or I with S; and a perpendicular to SL at S intersects LC perpendicular to the curve in the center of curvature. When Fig. 170.

FIG. 169. AL is an equilateral hyperbola, the angular velocity of the tangent at L, or of the perpendicular SP, is equal to the angular velocity of SL, (by what was ſhewn at the end of art. 315.) and LC is equal to LV, or LI to LS; but they are on oppoſite ſides of the point L. When the angular velocity of SL is to the angular velocity of SP in an invariable ratio, the angular velocities of SP and ST are likewiſe in an invariable ratio; and PM is to PK, or SL, in the ſame ratio. For example, when AL is a circle, and S is in the circumference, PM is two thirds of SL. In this caſe P is in an epicycloid that is deſcribed by a point in the circumference of a circle while it revolves on an equal circle; and M is in an epicycloid of the ſame kind.

FIG. 171. 392. COR. III. Let AE be a right line given in poſition, SA a perpendicular to it in A from the given point S; and let any other right line from S meet it in M; let the angle ASL be always to the angle ASM in any invariable ratio expreſſed by that of n to unit; and, SA, SM being the two firſt terms of a geometrical progreſſion, let SL be equal to the term of this progreſſion whoſe place in the ſeries is denoted by $n+1$; or, more generally, let SL be to SA as the power of SM, whoſe exponent is any poſitive number n, is to the ſame power of SA: then the angle SLP (contained by SL and the tangent at L) ſhall be equal to the angle SMA; the ray of curvature LC ſhall be to LV as n is to $n-1$; and, if SB be to SA as $n-1$ is to $n+1$, the variation of curvature at L ſhall be as AM directly and as SB inverſely. For, let circles deſcribed from the center S through M and L meet SA in F and f; and, the points F, f remaining fixed while M and L are ſuppoſed to proceed in the lines AM and AL, the fluxion of the arch FM ſhall be to the fluxion of fL in the ratio compounded of that of SM to SL and that of the fluxion of the angle ASM to the fluxion of ASL, (or of unit to n;) the fluxion of SM is to the fluxion of SL in the ſame ratio, by art. 167. conſequently, the fluxion of SL is to the fluxion of SM as the fluxion of the arch fL is to the fluxion of FM. Therefore the angle SLP is equal to SMA, by prop. 16. and if SP be perpendicular to LP, the angle ASL ſhall be to ASP in the invariable ratio of n to $n-1$. The fluxion of the angle ASL is to the fluxion of ASP, and (by

(by this propoſition) the ray of curvature LC is to LV, in the ſame ratio. The fluxion of the ray of curvature LC is to the fluxion of LV as n is to $n-1$, (art. 24.) the fluxion of LV is to the fluxion of SL in the ratio compounded of that of $n+1$ to n and that of LV to SL, (becauſe LV is to SL as SM is to SA, or as the power of SL whoſe exponent is $1+\frac{1}{n}$ is to the ſame power of SA;) and the fluxion of SL is to the fluxion of the curve AL as LP is to SL, or AM to SM. Therefore the fluxion of the ray of curvature LC is to the fluxion of the curve AL as AM is to SB; and the variation of curvature (as it is underſtood by Sir ISAAC NEWTON, and was explained in art. 369.) is as AM directly and SB inverſely. Hence, the variation of curvature at L in any given figure of this kind is as the tangent of the angle contained by SL and the perpendicular to the curve, and in any of thoſe figures is to the variation at the point in the parabola where the right line from the focus interſects the curve in an angle equal to SLP, as SA is to 3SB.

In the logarithmic ſpiral AL the fluxion of the ray of curvature is to the fluxion of the curve as SC is to SL, or LP is to SP; and, conſequently, in the invariable ratio of the tangent of the angle LSP to the radius. Therefore the variation of curvature in this figure is invariable, (as appears likewiſe from art. 370.) and is to the variation of curvature at the point of a parabola where the right line from the focus interſects the curve in an angle equal to the given angle SLP as 1 is to 3. FIG. 170.

393. COR. IV. If we ſubſtitute a ſemicircle AMS in the place of the right line AE in the conſtruction of the laſt article, it will appear in the ſame manner that the angle LSP is equal to ASM, (SP being now the ſame way from SL that SM is from SA,) that the ray of curvature LC is to LV as n is to $n+1$, and that (SM being produced till it meet Am perpendicular to SA in m and SB being taken to SA as $n+1$ is to $n-1$) the variation of curvature at any point L is as Am directly and SB inverſely, and in any given figure of this kind is as AM or the tangent contained by SL and the perpendicular to the curve. FIG. 172.

394. COR. V. If any curve AM be ſubſtituted for the right line AM in cor. 3. the reſt of the conſtruction remaining, it will FIG. 173.

appear in like manner that the angle SLP contained by SL and LP the tangent of AL shall be equal to the angle SMT contained by SM and the tangent of AM. Let Mc, LC be the rays of curvature at M and L, respectively; let S*u*, S*v* perpendicular to SM and SL meet those rays in *u* and V; and let *cr* be to *su* as unit is to *n*: then LC shall be to LV as M*c* is to M*r*. The demonstration is easily deduced from the equality of the angles SLP, SMT, by this proposition. Thus, if AM be a conic section, S the center, SA half the transverse axis, the angle ASL equal to 2ASM, and SA, SM, SL be in continued proportion; then AL shall be a conic section that shall have its focus in S, and, *cu* being bisected in *r*, LC shall be to LV as M*c* is to M*r*.

FIG. 123. 395. COR. VI. Let CP and SP revolve about the poles C and S with any angular velocities that are to each other in the invariable ratio of ST to CT, as in art. 315. & 316. make the angle CPN equal to SPT and TP*l* equal to TPS, and let P*l* meet CS in L; let PO perpendicular to PN meet CV perpendicular to CP in V, and let PG be to PV in the invariable ratio of ST to CT: then, if PO be the ray of curvature of the line described by P, PO shall be to PG as TL is to the sum or difference of TL and ST according to the different positions of the points T and L with respect to the poles C and S. The demonstration is deduced from prop. 18. and art. 315. by this proposition. If SMC be an arch of a circle, the arch SZ be to SM in any given ratio, and CZ always meet the right line SM in P, the tangent of the line described by the point P is determined by art. 315. and its curvature by this construction; because the angular velocity of P is to the angular velocity of CM, or of SP, in the same given ratio.

396. The following proposition is the sequel of the 18th and 26th, and is of use in enquiries concerning the curvature of lines that are described by means of right lines revolving about given poles, or of angles that either revolve about such poles or are carried along fixed lines, in the manner explained by several examples in the preceeding chapter from art. 315. to art. 324.

PROP.

Fig. 166. N.1.

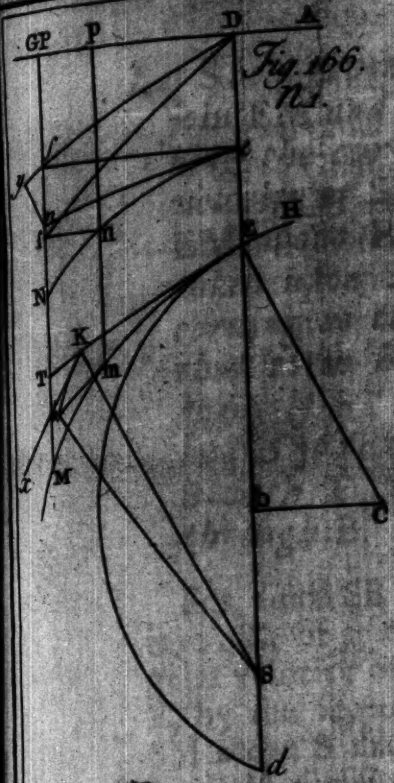


Fig. 166. N.2.

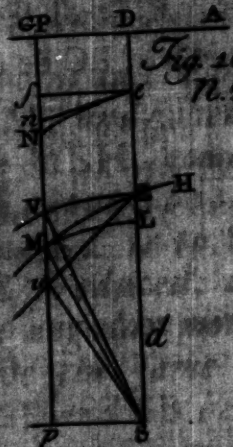


Fig. 167.

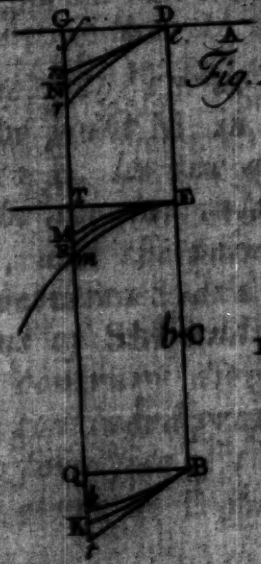


Fig. 169. N.1.

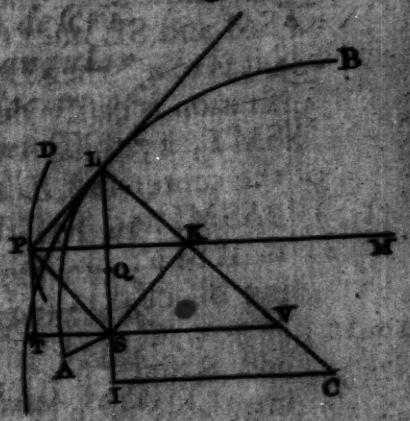


Fig. 168.

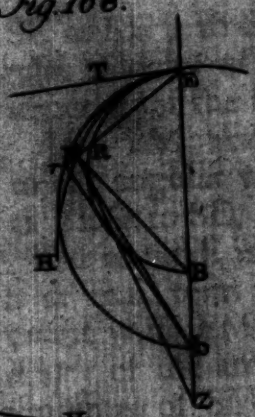


Fig. 169. N.2.

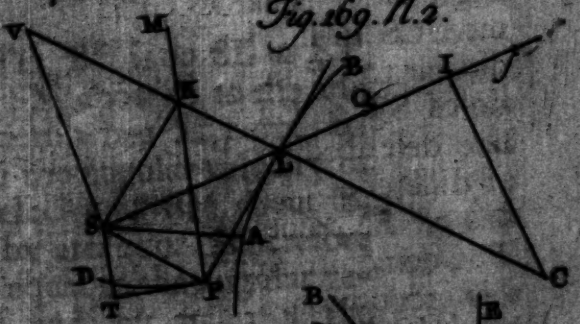


Fig. 170.

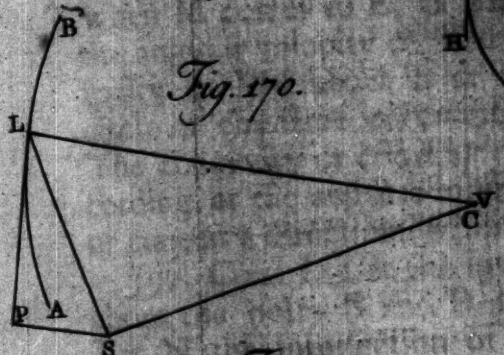


Fig. 171.

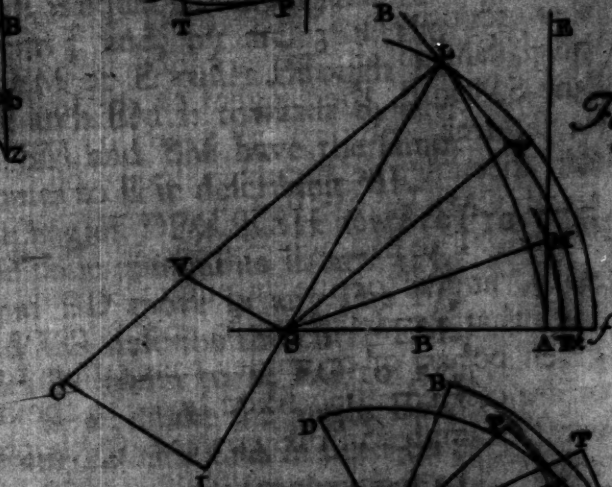


Fig. 172.

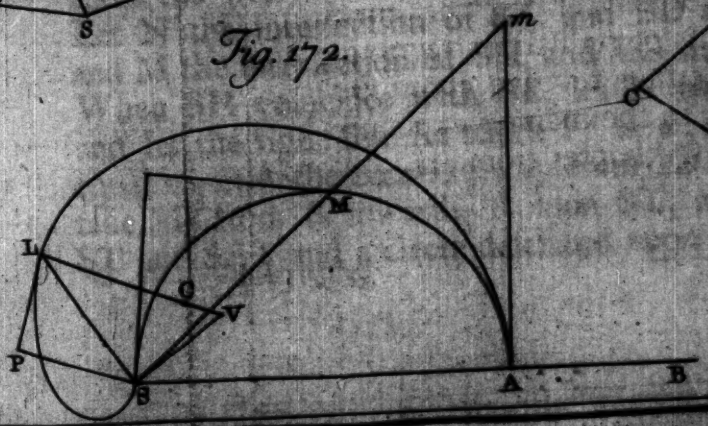
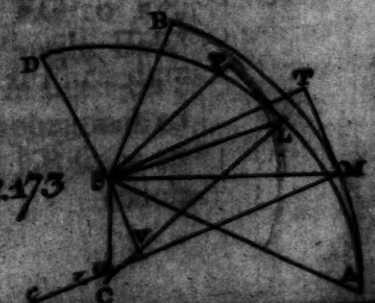
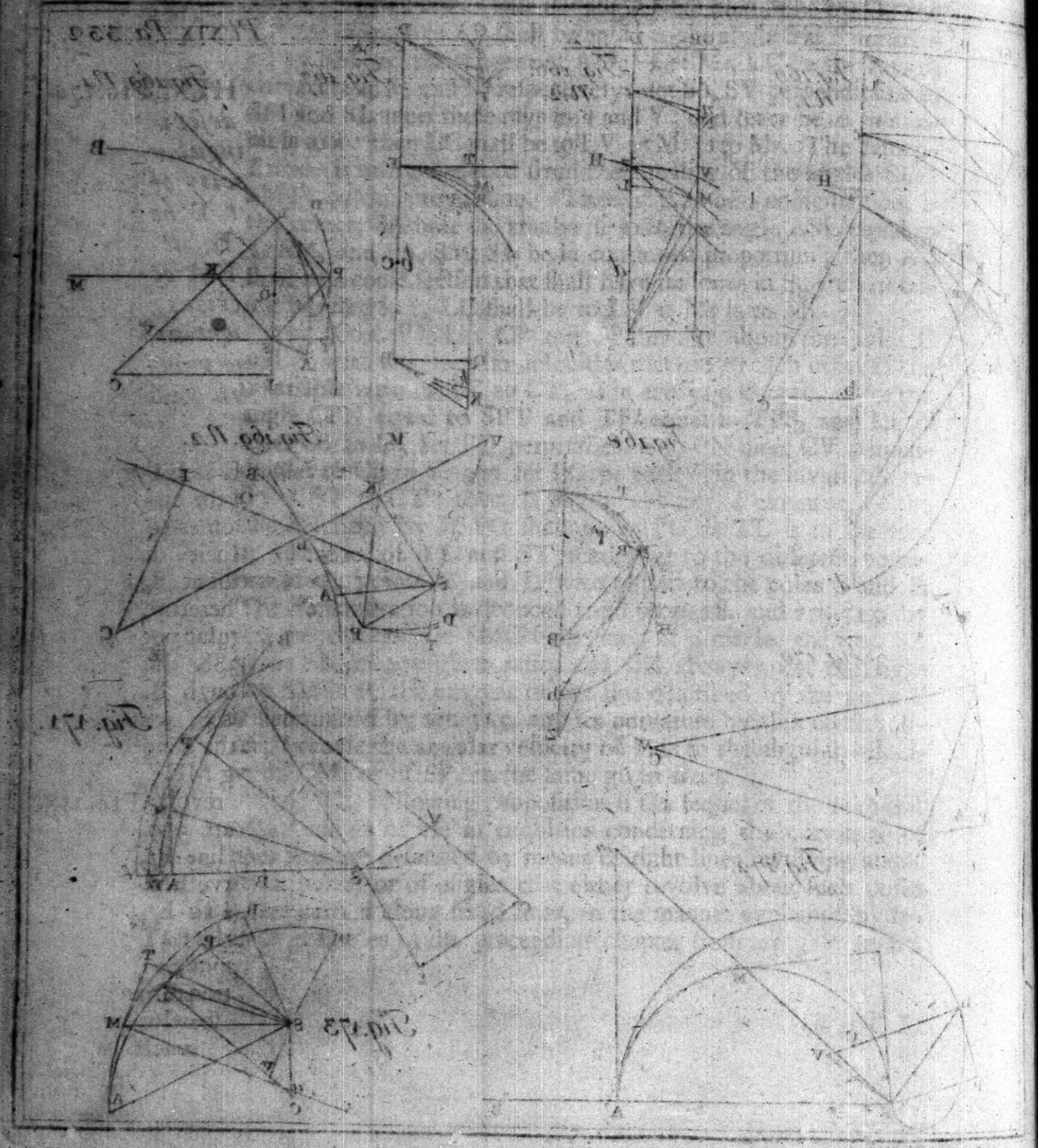


Fig. 173





PROP. XXXV.

Let S *be a given point in the plane of the curve* HEM, *join* EM *and* SM, *and when the point* M, *in describing the curve* MEH, *comes to* E, *let the angular velocity of* SM *about* S *be to the angular velocity of* EM *about* E *as* ET *is to* ST, *and the point* T *be taken upon* SE *produced, or between* S *and* E, *according as the angular motions of* SM *and* EM *at that term have the same or contrary directions; then, if* ST, SE *and* SB *be in continued proportion, and* B *be taken upon* SE *on the same side of the point* S *with* T, *the circle of curvature at* E *shall pass through* B. FIG. 174.

For, since SB is to SE as SE is to ST, BE is to SE as ET is to ST, or as the angular velocity of SM about S to the angular velocity of EM about E at the term when M comes to E, by the supposition. Join BM, and the angular velocity of SM about S shall be to the angular velocity of BM about B, when M comes to E, as BE is to SE, by art. 209. Therefore the angular velocity of EM about E is equal to the angular velocity of BM about B at that term; and, by art. 388. the circle of the same curvature with EM at E passes through B. The concavity or convexity of the arch EM is towards S, according as the angular motions of SM and BM have the same or contrary directions while M comes to E in describing ME.

397. COR. I. Let the invariable angles DEG, KSH revolve about the poles E and S after the same manner as in art. 319. Let N the intersection of SK and ED move in any line Ff, and M the intersection of SH and EG describe the curve EM. When SH coincides with SE, let N come to n, EG to Eg, and let the right line An touch Ff in n; make the angle EnT equal to SnA the contrary way from En that SnA is from Sn; take SB from S towards T upon ST, a third proportional to ST and SE, and a circle described upon the chord EB so as to touch FIG. 175.

touch Eg ſhall be the circle of the ſame curvature with EM at E. The curvature at S is determined in the ſame manner; and, if SK touch the curve Ff at *u* when EG coincides with ES, then S ſhall be a point of contrary flexure in the curve EMS; and, if Ff has a continued curvature at *u*, the curvature of EMS at S ſhall be leſs than that of any circle. When the right line S*n* is itſelf the tangent of Ff at *n*, and Ff has a continued curvature at *n*, the point E is a cuſpid, and the curvature at E is greater than in any circle.

FIG. 176. 398. COR. II. Suppoſe that it is required to deſcribe a conic ſection through the points S and M, that ſhall touch a given right line Eg at E ſo that the circle of curvature at E ſhall meet SE in B. Let SA be taken from S the ſame way with SB, ſo that SB, SE and SA may be in continued proportion; let the angle SEg revolve about E; and, when the ſide Eg comes to EM, and ES to E*d*, let SM interſect E*d* in N, join AN; let the right line MN revolve about the point S, and its interſection with the ſide EN move over the right line AN, then ſhall its interſection with the other ſide EM deſcribe the conic ſection required.

FIG. 177. 399. COR. III. In the ſame manner, if it is required to deſcribe a line of the third order through the double point S, the three points M, C and K ſo as to touch Eg in E, and EB the chord of the circle of curvature at E be given; let the point A be determined as in the laſt problem; let the angle SEg revolve about E and a right line MN about S, in the ſame manner; find three points N, *c* and *k* from M, C and K as N was found from M in the laſt problem, deſcribe a conic ſection through S, A, N, *c* and *k*; and, if the interſection N move always in this conic ſection, M ſhall deſcribe the line of the third order required. If it be required that the line ſhall have a point of contrary flexure at E, the conic ſection is to be deſcribed through N, *c* and *k* ſo as to touch SE at S.

FIG. 136. 400. COR. IV. The five points A, B, C, S and E in a conic ſection being given, let it be required to determine the circle of curvature at C. Determine the points D, *q*, *n* and the tangent C*n* as in art. 324. make the angle D*qr* equal to S*q*A the contrary way from D*q* that *q*A is from S*q*, and let *qr* meet SD in *r*; make the angle D*nx* equal to C*n*A with the like precaution,

tion, and let nx meet CD in x; join rx, and let it meet CS in T; let ST, SC and SB be in continued proportion, and the points B and T be on the same side of S: then a circle described upon the chord CB so as to touch Cn shall be the circle of curvature at C. There arise various constructions for determining the curvature of lines of the higher orders from this proposition, analogous to those by which the tangents and asymptotes of lines were determined in the last chapter. But, instead of insisting on these, it will be more worth while to add here a property of the lines of the third order to those that have been observed by Sir ISAAC NEWTON, *Enumer. linear. tertii ordinis.*

401. Let A be any point in a line of the third order from which two tangents AC, AS can be drawn to the curve in C and S; from any point in the curve, as P, draw right lines to C and S that meet the curve again in M and N, respectively; join CN and SM, and the intersection of these lines shall be always in the curve. If CN and SM be parallel to each other, they shall be parallel to the asymptote of an hyperbolic branch, or shall shew the position to which the tangent of a parabolic branch continually approaches while the figure is produced; that is, according to the usual stile, they shall intersect each other in the curve either at a finite or at an infinite distance: And if two other tangents Ac, Af can be drawn from the same point A to the curve in c and f, the right lines that join any of the points of contact C, S, c and f shall intersect each other in the curve. We may explain this property, with its consequences, more fully on another occasion, and shew how a line of the third order (whether it have a *double* point, or not) can be described through seven points so as to touch two right lines given in position at two of those points; and shall only observe further, to illustrate this property, that it holds of any conic section and right line described in the same plane: for if CA and SA touch a conic section in C and S, and from any point P the right lines PC, PS be drawn that meet the conic section in M and N; then CN and SM shall always intersect each other in some point of the right line AP. Of this see Mr. SIMSON's *Sect. conic. lib. 5. prop. 45.* But to return to the curvature of lines:

FIG. 178.

FIG. 179.

402. A

FIG. 180. 402. A flexible line or thread IC*a*A being applied along the convexity of the curve IC*a* from I to *a*, let the part *a*A be extended in a right line that touches the curve in *a*; then, while one extremity of the line, or thread, remains fixed at I, let the other extremity A move towards H, so that the line may be gradually separated from the curve, and the part CE which is not applied to it be always extended in a right line that touches the curve; then the point E shall in this manner trace a curve AEM that by the excellent Mr. HUYGENS is said to be described by the evolution of *a*CI, which is itself called the *evoluta*.

PROP. XXXVI.

Let the right line CE *touch the evoluta a*CI *in* C, *and meet the curve* AEM, *that is described by the evolution of a*CI, *in* E; *then a circle* ERB *described from the center* C *through the point* E *shall have the same curvature with* AEM *at* E: *and if c be the center of the circle of the same curvature with the evoluta a*CI *at* C, *the variation of the curvature of* AEH *at* E *shall be measured by the tangent of the angle* CE*c*.

Let the right line QM touch the evoluta at any point Q betwixt C and I, and meet the curve AEH, the circle ERB and right line EB in M, R and N; join CR, and, since the right line QM is equal to the arch QC and right line CE (or CR) taken together, QM is greater than QR; consequently, the arch EM of the curve EMH passes without the circle ERB, and it is manifest that no circle described through E with a radius less than CE can pass between EM and ER. Because the sum of QN and NC is greater than the arch QC, (art. 183.) the sum of QN and NE is greater than the sum of the arch QC and ray CE, or QM; therefore NE is greater than NM: and a circle described from any point N through E, with a radius NE greater than CE, passes without EM and ER. Therefore no circle can pass between EM and ER, and ER is the circle

of the ſame curvature with EM at E, by art. 364. In like manner it is ſhewn, that the circle RE produced on the other ſide of E paſſes without EA, and that no circle can be drawn between them on that ſide. It is evident, that the curvature decreaſes from E towards H, but increaſes from E towards A; and that the curve paſſes within or without the circle of curvature, according as the curvature increaſes or decreaſes. The variation of curvature at E is meaſured by the ratio of the fluxion of the ray of curvature CE (or of the curve *a*C) to the fluxion of the curve AE, or (becauſe the angular motion of CE the tangent of *a*CI is equal to the angular motion of *c*C about *c*, by art. 388.) by the ratio of C*c* to EC, that is, by the tangent of the angle *c*EC.

403. Cor. I. The length of the arch CQ is equal to the difference of the rays of curvature MQ and EC, when the curvature increaſes or decreaſes continually from E to M. Hence, from any geometrical curve another may be deduced that ſhall admit of an accurate rectification. If we ſuppoſe, (as in art. 366.) that TMK is always parallel to EC, the rectangle MTK equal to the ſquare of ET, and BV the tangent of BKF at B to meet TE in V, then ſhall EV, EC and one third part of C*c* the ray of curvature of the evoluta *a*CI be in continued proportion. For, (by art. 386.) when K ſets out from B, the fluxion of TK is to the fluxion of the ray of curvature EC as 2 is to 3; conſequently, BE is to EV as 2C*c* is to 3EG: but BE is equal to 2EC, (by prop. 32.) therefore EV is to EC as EC is to one third part of C*c*; and if B*r* perpendicular to the curve BKF meet ET in *r*, C*c* ſhall be equal to three fourths of E*r*. Thus the curve BKF, by which the curvature of EMH and its variation were determined in prop. 32. and its corollaries, ſerves likewiſe for determining the curvature of the evoluta *a*CI.

404. Cor. II. Hence a ready way is deduced for finding the center of curvature of the line by whoſe evolution any conic ſection is deſcribed. Let E be any point in a conic ſection, EO a diameter through E, *b* the center of curvature at E; let *b*Y parallel to the tangent ET meet EO in Y; take *bc* equal to 3EY upon Y*b* produced from *b*, and *c* ſhall be the center of curvature of the line by whoſe evolution the conic ſection EMH is deſcri- FIG. 160.

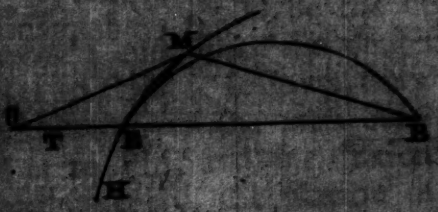
described. For the triangles VE*b*, E*b*Y are similar, (by the 4th property of the circle of curvature, art. 375.) and VE is to E*b* as E*b* is to *b*Y, and *c* is the center of curvature belonging to the point *b* of the evoluta, by the last corollary. When E is in the extremity of either axis, *b*Y and consequently *bc* vanish, the evoluta has a cuspid where the curvature is such as
FIG. 171. was described in art. 378. In like manner, in art. 392. let CY parallel to the tangent LP meet LS in Y, let C*c* be to CY as $n+1$ is to $n-1$, and *c* shall be the center of curvature of the line *c*C*z* by whose evolution LA may be described. And by a similar construction the center *c* is determined in art. 393.

405. But to proceed now to consider some of the useful pro-
FIG. 181. blems that have a dependence on the curvature of lines: When a circle moves upon a given right line so as always to touch this right line and apply the parts of its circumference successively to it, any given point in the circumference describes a *Cycloid*. Let IV be the given right line, and C the given point in the circumference of the circle KCL touch that right line first in I, and again in *i* after describing the curve ICA*i* while the circle makes a compleat revolution; bisect I*i* in B, and let BA perpendicular to I*i* meet the curve in A; then ICA*i* is the cycloid, I*i* its base, AB its axis, and AMB the generating circle. If the circle proceed still in the same manner along the right line IV, the same given point of the circumference will describe an equal and similar figure in every revolution of the generating circle; from which it is obvious, that this figure is not of those that are called geometrical, which can never meet a right line in more than a certain definite number of points; for it may be continued till it meet the base I*i*, or any right line parallel to I*i* betwixt I*i* and A*b*, in any assignable number of points. It has however several remarkable properties, useful in philosophy; some of which we shall briefly demonstrate.

Any ordinate from the cycloid, as CP, perpendicular to the axis in P that meets the semicircle AMB in M is equal to the sum of the arch AM and of its right sine MP. For, let the generating circle KCL touch the base in K when the point that describes the cycloid comes to C; then, since the arch KC or BM is equal to the right line KI, and the semicircle BMA

Fig. 174. N. 1

Fig. 174. N. 2.



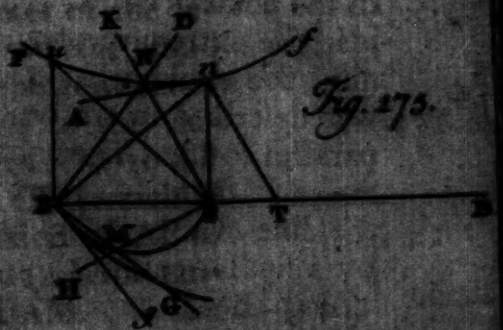


Fig. 175.

Fig. 176.

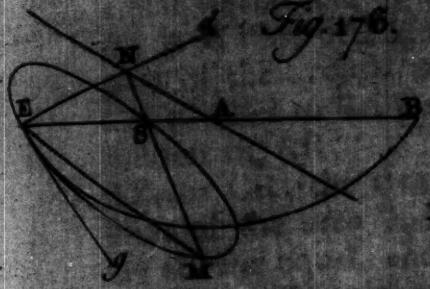


Fig. 177.

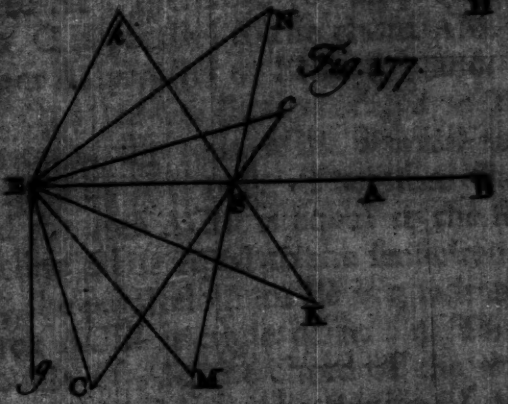


Fig. 180.

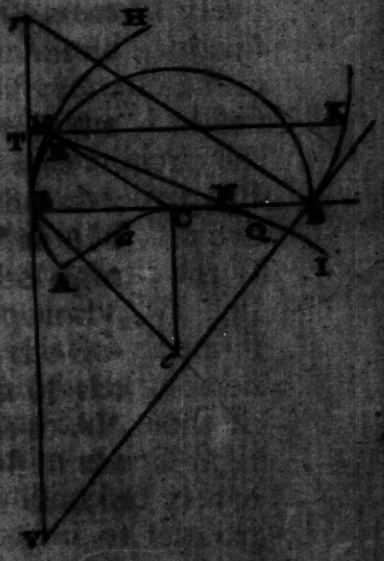
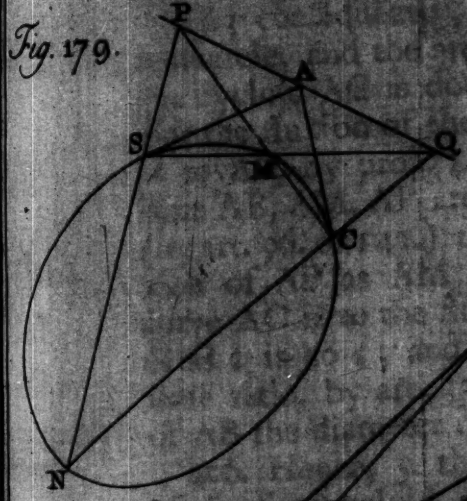


Fig. 179.



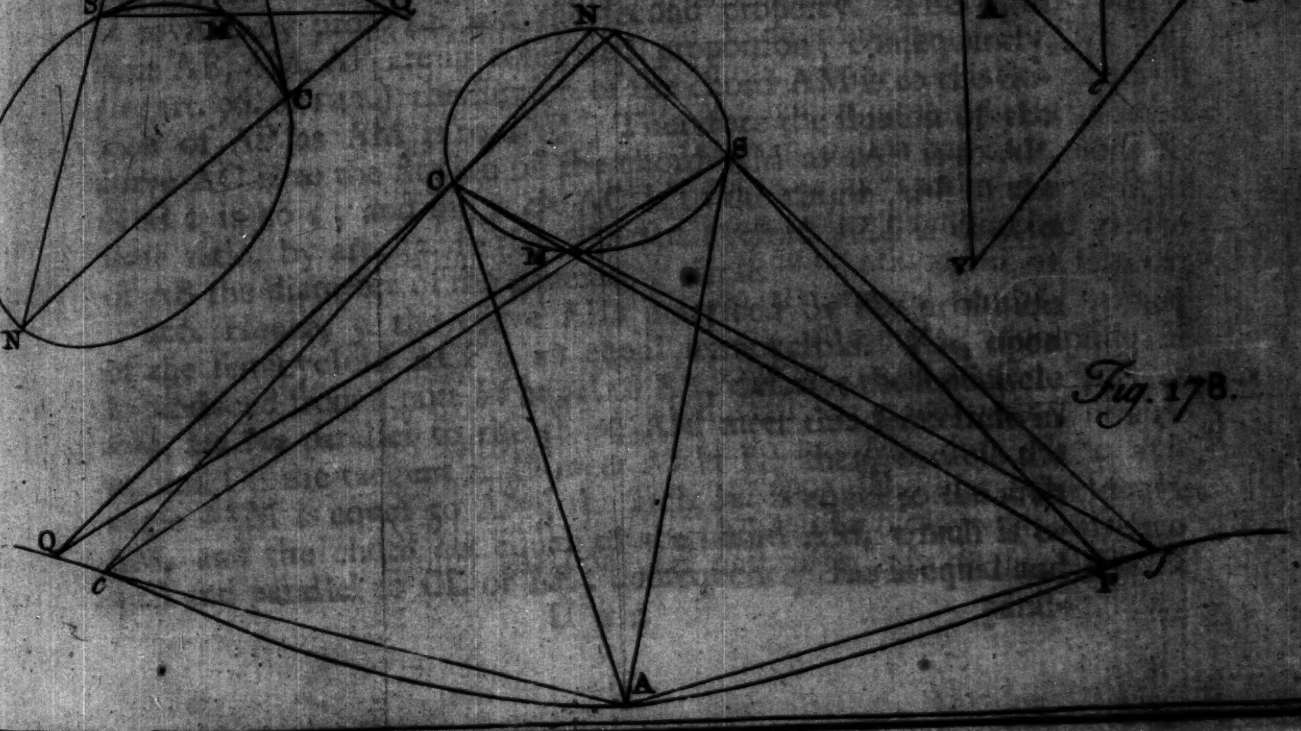
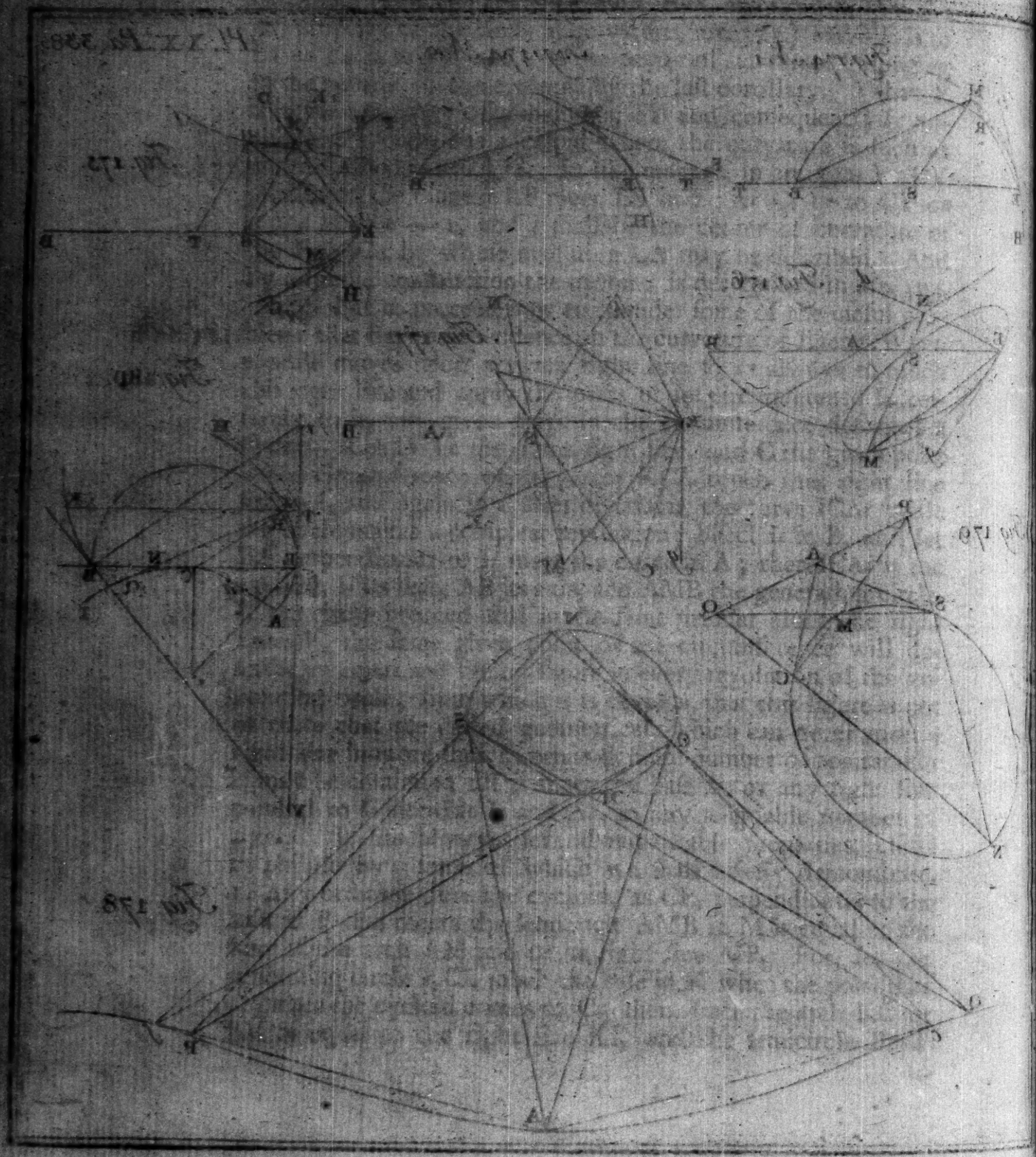


Fig. 178.



to BI, (by the deſcription,) BK, or MC, is equal to the arch CL, or AM; and CP is equal to the ſum of the arch AM and its right ſine MP. 2. The tangent of the cycloid at C is parallel to the chord AM. For, let O be the center of the ſemicircle AMB, join OM; and, ſince the tangent of the circle at M is perpendicular to OM, it follows from the fourteenth propoſition, that the fluxion of the arch AM is to the fluxion of AP as OM is to PM, and the fluxion of PM to AP as OP is to PM; conſequently, the fluxion of CP (which is equal to the ſum of AM and PM) is to the fluxion of AP as BP is to PM, or PM to AP: therefore (by prop. 14.) the tangent of the cycloid at C is parallel to the chord AM. 3. Let CR parallel to BA meet A*b* parallel to BI in R; and, ſince the fluxion of AR (or PC) is to the fluxion of AP as PM is to AP, or CR, the fluxion of the area ACR is equal to the fluxion of the area A*x*MP, by art. 111. Therefore theſe areas are always equal, and the area ACI*b* is equal to the ſemicircle A*x*MBA. The right line BI is equal to the ſemicircumference AMB; conſequently, the parallelogram B*b* is quadruple of the ſemicircle, and the area ACIB is triple of it. 4. The arch of the cycloid AC is double of AM the chord of the arch A*x*M. For the fluxion of the arch AC is to the fluxion of AP as AM is to AP, by prop. 14. and the ſecond property. The right lines AB, AM, AP are in continued proportion; conſequently, (by art. 96. or 142.) the fluxion of the chord AM is to the fluxion of AP as AM is to 2AP. Therefore the fluxion of the curve AC is to the fluxion of the chord AM as 2AP is to AP, or as 2 is to 1; and the arch AC is to the chord AM in the ſame ratio, by art. 33. ſo that the ſemicycloid ACI is double of AB the diameter of the generating circle.

406. Hence, 5. the curve AE*a* deſcribed by the evolution of the ſemicycloid ACI is an equal ſemicycloid. For, upon I*b* produced from *b* take *ba* equal to BA, deſcribe the ſemicircle *bma*, let *bm* parallel to the chord AM meet this ſemicircle in *m*, and CE the tangent at C meet A*b* in L; then, becauſe the angle *b*AM is equal to A*bm*, the arch *bm* is equal to the arch AM, and the chord *bm* equal to the chord AM, which is equal and parallel to CL or LE; conſequently, E*m* is equal and

parallel to *b*L or KI, which is equal to the arch BM or *am*; and E*p* the ordinate from E perpendicular to *ba* is equal to the sum of the arch *am* and of its right line *mp*. Therefore, by the converse of the first property, AE*a* is a semicycloid that has *ba* for its axis and *bma* for its generating semicircle. Hence the ray of curvature of the cycloid AE*a* at E is equal to 2*bm*; and the variation of curvature is measured by the tangent of the angle *abm*, or of the angle contained by the tangent at E and the ordinate E*p*, by art. 402. It appears likewise, that a heavy body may be made to describe any arch of a cycloid E*a* by suspending it from a flexible line or thread ICE equal to 2*ab*, that has one end fixed at I and is applied to the convexity of the semicycloid ICA from I to C, I*b* being perpendicular to the horizon: for E*a* will be described by the evolution of IC; and, by an equal and similar semicycloid touching I*b* on the other side at I, a pendulum may be made to oscillate in this figure.

407. If the motion of the generating circle KCL upon the base IB, or of the point M in the semicircle BMA, be uniform, and its velocity be the half of that which would be generated by an uniform gravity by falling through the diameter BA, then, 6. the velocity of C in describing the semicycloid ICA will be equal to that which would be acquired by the same gravity by falling through BP, or by falling along the arch of the cycloid IC from I, BA being supposed perpendicular to the horizon. For, since the arch AC is double of the chord AM, the fluxion of the arch AM is to half the fluxion of the arch AC (or the velocity of M in the semicircle is to half the velocity of C in the semicycloid) as AB is to BM, (by prop. 17.) or in the subduplicate ratio of AB to BP; and the velocities acquired by an uniform gravity by falling through AB and BP are in the same ratio, by art. 95. Hence, the time in which a heavy body describes a semicycloid ICA, or AE*a*, by descending along it by its gravity, is to the time in which it would fall through BA the diameter of the generating circle by the same gravity as the semicircumference of a circle is to its diameter. For that time is the same in which the point K describes IB, or the point M describes the semicircumference BMA, by an uniform motion

tion with half the velocity that would be acquired by falling through BA; and the time in which a body falls from B to A by the ſame gravity is equal to the time in which it would deſcribe BA with the ſame uniform motion, by art. 95.

408. In general, 7. let a body deſcend by its gravity from any point in the cycloid AE*a*, as H, along the curve HE*a*, the axis *ba* being perpendicular to the horizon; let the ordinate from H meet the axis *ba* in *d*; upon the diameter *da* deſcribe the ſemicircle *dna*; and let E*p* the ordinate from the body at any point E always meet this ſemicircle in *n*: then ſhall the motion of the point *n* in the ſemicircle *anb* be uniform ſo as to meaſure the time, and its velocity ſhall be equal to the half of that which would be acquired by falling through the right line *fa* a third proportional to *ba* and *da*. For the chord *an* is always to the chord *am* in the ſubduplicate ratio of *da* to *ba*, and the fluxion of the chord *an* to the fluxion of the chord *am* (or one half of the fluxion of *a*E by the fourth property) in the ſame ratio. The fluxion of the arch *an* is to the fluxion of the chord *an* (by prop. 16.) as *ad* is to *dn*, or in the ſubduplicate ratio of *ad* to *dp*. Therefore the fluxion of the arch *an* is to half the fluxion of *a*E in the ſubduplicate ratio of the ſquare of *ba* to the rectangle contained by *ba* and *dp*, that is, in the ſubduplicate ratio of *fa* to *dp*; and the velocities acquired by falling through *fa* and *dp* are in the ſame ratio. But the velocity in the cycloid at E is equal to that which would be acquired by falling from *d* through *dp*; conſequently, the velocity of *n* in the ſemicircle *dna* is half of that which would be acquired by falling through *fa*. Hence, the time in which the body E deſcribes any arch of the cycloid HE is the ſame in which the point *n* deſcribes the arch *dn* by an uniform motion with half the velocity that would be acquired by falling through the perpendicular *fa*; and the time in which the body E deſcends along the arch H*a* by falling from H to *a*, is equal to the time in which the point *n* with the ſame uniform motion deſcribes the ſemicircle *dna*, which is equal to the time in which the point *m* deſcribes the ſemicircle *bma* uniformly with half the velocity acquired by falling through *ba*, becauſe *dna* is to *bma* as *da* is to *ba*, or in the ſubduplicate ratio of *fa* to *ba*;

that

that is, in the ratio of the velocity acquired by falling through *fa* to that acquired by falling through *ba*. Therefore the time in which a body deſcends along the arch H*a* from any point H to the lowermoſt point *a* is equal to the time in the ſemicycloid A*Ea*, and is the ſame where-ever the point H be in A*Ea* from which it begins to deſcend, which is the celebrated property of the cycloid diſcovered by Mr. HUYGENS. The ſame property may be demonſtrated by ſhewing, that the power by which the velocity of the body in the cycloid is accelerated at E is always as E*a* the arch from E to the lowermoſt point *a*; and an analogous property is ſhewn of the epicycloid (which is deſcribed by a given point in the plane of a circle that revolves upon a circular baſe) by Sir ISAAC NEWTON. Produce *an* till it meet the circle *bma* n *v*; and, if *ve* parallel to *n*E meet the cycloid in *e*, the arch HE ſhall be deſcribed when the motion begins from H in the ſame time that A*e* is deſcribed when the motion begins from A.

409. Rays of light being ſuppoſed to iſſue from a given point, and to be reflected by a given curve, ſo as to make the angle of reflexion equal to the angle of incidence, a curve that touches all the reflected rays is called the *Cauſtic by Reflexion*. Let S be the given point from which the rays iſſue, (which is therefore called the *focus* of the incident rays,) SL any incident ray, PL*p* the tangent at L, LC the ray of curvature at L, L*m* the reflected ray conſtituting the angle CL*m* equal to CLS; then, if the reflected rays always touch the curve *bme*, it is the cauſtic by reflexion. Let SP perpendicular to the tangent LP meet it always in P a point of the curve DP; let HME be the curve by the evolution of which DP is deſcribed according to art. 402. and let PM touch HME in M; join SM, and produce it to *m*, ſo that S*m* be equal to 2SM: then *m* ſhall be a point in the cauſtic of the curve BL, when S is the radiating point. For, becauſe MP is perpendicular to the curve DP, the angle MPS is equal to the complement of SLP (by art. 211. & 212.) or to LSP, and SL is biſected by MP in K; therefore S*m* is to SM as SL is to SK, L*m* is parallel to KM the tangent of HME: but the figure *bme* is ſimilar to HME, and ſimilarly ſituated, (art. 122.) therefore L*m* is the tangent of *bme*. Becauſe L*m* is parallel to PM, the angle *m*LC is equal to MPS, or LSP, or CLS;

Fig. 182.

CLS; therefore, when SL is the incident ray, L*m* is the reflected ray, and *m* is a point in the cauſtic.

410. Let L*ſ* be taken on the reflected ray eqpal to LS; and, CR being perpendicular from the center of curvature on this ray in R, biſect LR in *q*; and *qſ*, *q*R and *qm* ſhall be in continued proportion. For, by art. 390. *ſq*, *ſ*L and 2PM are in continued proportion. Let CI be perpendicular to SL in I, and IL be biſected in Q; when S is on the concave ſide of the curve, and LS is greater than LQ, PM is equal to the ſum of PK (or SK) and KM, and 2PM is equal to the ſum of *ſ*L and L*m*; therefore *ſq* is to *q*L or *q*R as *ſ*L is to L*m*, or as *q*R is to *qm*. When S is betwixt Q and L, 2PM is equal to the difference of L*m* and L*ſ*, and *ſq* is to *q*L as *ſ*L is to L*m*, or as *q*L (or *q*R) is to *qm*. In like manner it appears, that when S is on the convex ſide of the curve, *ſq*, *q*R (or *q*L) and *qm* are in continued proportion. When the incident ray is perpendicular to the curve, the reflected ray coincides with the incident ray, *ſ* with S, C with R, and *qm*, *q*C, *q*S are in continued proportion. In general, the rectangle *ſqm* is equal to the ſquare of *q*R, or *q*L; and, when the incident rays are parallel, the point *m* muſt coincide with *q*, and L*m* be equal to one half of LR. When LS is equal to one half of LI, and S is on the concave ſide of the curve, *ſ* coincides with *q*, and the reflected ray L*m* becomes an aſymptote of the cauſtic. Fig. 183.

411. Let BL be a circle as in art. 283. C the center, CB the radius that paſſes through S the radiating point; biſect BC in *q*, let *q*S, *q*C and *q*H be in continued proportion: and, when L comes to B, *m* ſhall come to H. When CS is leſs than one half of CB, the curve DP has no point of contrary flexure, and the cauſtic has no aſymptote. When CS is equal to one half of CB, the diameter through S is the aſymptote of the cauſtic. When CS is greater than one half of CB, but leſs than CB, the cauſtic meets CB produced beyond B, the part of the curve DP adjoining to B is convex towards S, P is a point of contrary flexure when SL is one fourth part of the chord LZ that paſſes through S, (as was ſhewn in art. 283.) and the reflected ray is then an aſymptote of the cauſtic. Fig. 105.

412. Let SA be a right line given in poſition, and L*m* ſhall be Fig. 182.

be to SL as the fluxion of the angle ASL is to the fluxion of 2ASP — ASL. Hence, in all the curves constructed in art. 392. & 393. LM is to SL in an invariable ratio, because in those figures the angle ASL is to ASP in an invariable ratio; only, in the parabola, S being the focus, the fluxion of ASL is equal to the fluxion of 2ASP, and the reflected rays are parallel to each other and to the axis of the figure.

413. When the rays that iſſue from a given point are refracted at the curve, ſo that the ſine of the angle contained by the refracted ray and the perpendicular to the curve is always to the ſine of the angle contained by the incident ray and that perpendicular, in one conſtant ratio, the curve that touches all the refracted rays is called the *Cauſtic by Refraction.* Let S be the radiating point, SL any incident ray, LR the refracted ray, C the center of the curvature at L, CI perpendicular from C on the incident ray in I, CR perpendicular on the refracted ray in R; join SC and IR; let RZ perpendicular to RI meet CI in Z; join LZ meeting SC in Q, and let QM parallel to RZ meet the refracted ray LR in M; then ſhall M be in the cauſtic. For, ſuppoſing the refracted ray LR to touch the cauſtic in any point M, and LT parallel to CI to meet SC in T, and the point L in deſcribing the curve BL to move towards *l*; then the angular velocity of SI about S ſhall be to the angular velocity of MR about M in the ratio compounded of the direct ratio of the fluxion of CI to the fluxion of CR (or the ratio of CI to CR, by art. 27. the ratio of CI the ſine of the angle of incidence to CR the ſine of the angle of refraction being ſuppoſed invariable) and of the inverſe ratio of SI to RM. The angular velocity of SL about S is to the angular velocity of ML about M (by art. 208. & 202.) in the direct ratio of LI to LR (or of CZ to CR, becauſe the triangles CZR, LIR are ſimilar) and the inverſe ratio of SL to LM. Therefore LM is to RM as the rectangle contained by CI and SL is to the rectangle contained by SI and CZ, or (becauſe SI is to SL as CI is to LT) as LT is to CZ, or LQ to ZQ; conſequently, QM and ZR are parallel, and the point M was rightly determined. When the incident rays are parallel, let CB parallel to thoſe rays through the center of curvature meet LZ

LZ in Q, and QM parallel to RZ shall meet the refracted ray LR in the caustic at M. When the incident ray becomes perpendicular to the refracting curve, that is, when L comes to B, the angular velocity of SL is to the angular velocity of ML as MB is to SB; therefore the caustic meets SC in M so that CM is to BM in the ratio compounded of the given ratio of CR (the sine of refraction) to CI (the sine of incidence) and of that of SC to SB: and when the incident rays are parallel, CM is to BM as the sine of refraction is to the sine of incidence. Fig. 185.

414. Describe the circle LICV upon the diameter LC, let CV parallel to the incident ray SI meet this circle in V; and RZ shall always pass through V, the center of curvature C and incident ray SL being given. Therefore the point M is determined by applying CR in this circle so that it may be to CI in the given ratio of the sine of refraction to the sine of incidence, joining VR that meets CI perpendicular from C to the incident ray in Z, joining LZ that meets SC in Q, and drawing QM parallel to VR. When LZ is parallel to SC, the refracted ray is the asymptote of the caustic. Fig. 184.

415. Let BL be a circle, and, the incident rays being supposed parallel, let the caustic touch the refracted ray LM in M a point of the circle; then, because LR is equal to RM, LQ and CI are bisected in Z and I; and, the triangles CRZ, LRI being similar, it follows, that CZ, or one half of CI, is to CR as LI is to LR. Therefore, if the invariable ratio of the sine of incidence to the sine of refraction be expressed by that of I to R, the square of I shall be to the square of 2R as the square of LI to the square of LR, and 4RR — II shall be to II as the difference of the squares of LR and LI, or of CI and CR, is to the square of LI; consequently, 3RR is to II — RR as the square of the radius CL is to the square of LI, or CV: And in this manner Sir Isaac Newton determines the position of the ray that, after a refraction at L, a reflexion at M, and a second refraction at G, defines the interior rainbow; for, the rays incident about L being refracted so as to touch the caustic near to M, a point in the circle BLM, and being thence reflected, they will emerge nearly parallel at G. If it be required that the rays refracted about L should be reflected at I in Fig. 186. Fig. 187.

the circle nearly parallel to each other; then, by art. 411. MR must be equal to Ml, or one half of LR, ZQ muſt be one half of LZ, CZ one half of ZI, and therefore equal to one third part of CI. But LI is to LR as CZ is to CR, or as I is to 3R, and 9RR — II is to II — RR as the ſquare of CI is to the ſquare of LI; therefore 8RR is to II — RR as the ſquare of the radius CL is to the ſquare of LI, or CV; and hence the ray is determined which after a refraction at L, two reflections at *l* and G, and a ſecond refraction at H, defines the exterior rainbow; for, if G*m* be one fourth part of GH, or of Ll, the rays reflected at G ſhall touch a cauſtic formed by this ſecond reflexion at *m*, and emerge parallel after their refraction at H.

416. Of the various problems that depend on the curvature of lines, none are more uſeful in philoſophy than thoſe which relate to the centripetal and centrifugal forces. In this doctrine it is ſuppoſed, that a body at reſt never moves of itſelf, and that a body in motion never changes the velocity or direction of its motion of itſelf; but that every motion would continue uniform and its direction rectilineal unleſs ſome external force or reſiſtance affected it. Hence, when a body at reſt always tends to move, or when the velocity of any rectilineal motion is accelerated continually, or when the direction of a motion is continually varied and a curve line deſcribed, theſe are ſuppoſed to proceed equally from the influence of ſome power that acts inceſſantly; which may be meaſured either by the preſſure of the quieſcent body againſt the obſtacle that hinders it to move in the firſt caſe, or by the acceleration of the motion in the ſecond, or by the flexure of the curve deſcribed in the third caſe, due regard being had to the time in which theſe effects are produced, and the other circumſtances, according to the principles of mechanics. Effects of the power of gravity of each kind fall under our conſtant obſervation near the ſurface of the earth; for the ſame power which renders bodies heavy while they are at reſt, accelerates them when they deſcend perpendicularly, and bends their motion into a curve line when they are projected in any other direction than that of their gravity. But we have acceſs to judge of the powers that act on the celeſtial bodies

by effects of the laſt kind only; and it is of theſe chiefly we are to treat here.

417. As the velocity of a variable motion is meaſured by the ſpace which would be deſcribed by it in a given time if it was continued uniformly for that time, (art. 4. 5. & 6.) and not by the ſpace that is actually deſcribed by the variable motion in that time; ſo the power by which the rectilineal motion of a given body is continually accelerated or retarded is meaſured by the increment or decrement of the velocity that would be generated by that power in a given time, if its action, or influence, was continued uniformly for that time, and not by the increment or decrement of the velocity that is actually generated if the action of the power varies. The fluxion of the velocity is meaſured in the ſame manner as was explained in art 70. Therefore the power that accelerates or retards a rectilineal motion is always meaſured by the fluxion of the velocity of the motion, or (becauſe the velocity is itſelf the firſt fluxion of the ſpace) by the ſecond fluxion of the ſpace deſcribed by the motion, the time being ſuppoſed to flow uniformly. Thus, when a given body aſcends, or deſcends, in the right line that is in the direction of its gravity, in ſpaces void of reſiſtance, the power that accelerates or retards its motion at any term of the time (that is, the accelerating force of its gravity) is meaſured by the fluxion of the velocity; and if the velocity increaſe or decreaſe uniformly ſo that its fluxion be conſtant, the gravity muſt be ſuppoſed uniform. If the body deſcend, or aſcend, in a medium that reſiſts its motion, the power that accelerates or retards its velocity (that is, the difference of the gravity and reſiſtance when the body deſcends, and their ſum when it aſcends) is ſtill meaſured by the fluxion of the velocity, or the ſecond fluxion of the ſpace that is deſcribed by the motion, the time being ſuppoſed to flow uniformly.

418. Suppoſe the right line Dd to move parallel to itſelf along the given line AO with an uniform motion, and the gravity to act always in the direction Dd; then the body will deſcend in this line Dd in the ſame manner as if the line was quieſcent and the gravity acted in its direction. Therefore the gravity will be ſtill meaſured by the fluxion of the velocity with Fig. 188.

with which the body descends in the right line Dd, or by the second fluxion of DE the space described by it in this line, that is, by the second fluxion of the ordinate of the curve traced by the body on the immoveable plane $AOo\alpha$, the time (or the right line AD which flows in the same manner as the time) being supposed to flow uniformly.

419. When a curve is described by a gravity that acts in parallel lines, its force is as the square of the velocity of the body directly and that chord of the circle of curvature which passes through the body in the direction of the gravity inversely. For, the same things being supposed as in the last article, let the velocity of the point D or the fluxion of AD be represented by a given line DG, let GH parallel to the ordinate DE meet the tangent ET in T, and the fluxion of the curve FE, or the velocity of the body in describing it, shall be measured by ET, by prop. 14. Let the circle of curvature at E meet Ed in B, and the second fluxion of DE shall be measured by a right line that is a third proportional to $\frac{1}{2}$EB and ET, by prop. 33. Therefore the gravity, which is measured by the second fluxion of DE, is as the square of ET directly and EB inversely, that is, as the square of the velocity directly and the chord of the circle of curvature which passes through E in the direction of the gravity inversely.

420. This may be further illustrated, if it seem necessary, in the following manner. If the gravity act uniformly and in parallel lines upon a body that sets out from E in the direction ET, and bend its course into the curve ER, and the right lines TR, $V\nu$ in the direction of the gravity meet ET in T and V, and meet the curve in R and ν, $V\nu$ shall be to TR as the square of EV is to the square of ET, by what was shewn in art. 95. consequently, the rectangle contained by $V\nu$ and an invariable right line is equal to the square of EV, and the curve $E\nu R$ is a parabola. And it may be shewn conversely, from art. 254. that when a parabola $E\nu R$ is described by a gravity that acts always in a direction parallel to the axis of the figure, then, DG being given, TR is invariable; and, consequently, the gravity is uniform. Let $HE\lambda$ be any curve line that touches the parabola $E\nu R$ and has the same curvature with it at E; then, if the power

by which EH is described act always in the same direction as the uniform gravity by which the parabola ER is described, and the velocities at E in EH and ER be equal, the gravity in the curve *b*EH at E must be equal to the uniform gravity in the parabola ER. For, if the gravity in EH at E be said to be greater than the uniform gravity in the parabola ER, let it be greater in the ratio of TZ to TR; and let EZ be a parabola described through E and Z that has the same tangent ET and diameter E*d* with the parabola ER at E. Then the gravity in the curve EH at E shall be equal to the uniform gravity in the parabola EZ; and, if the variable gravity in EH be supposed first to increase while the body moves from E to H, TZ shall be always less than TH, because a space described by a power that acts uniformly must be less than the space which is described in the same time when that power continually increases from the beginning of the time. But TR is always less than TZ; therefore the parabola EZ passes between the curve EH and parabola ER. But EH and ER were supposed to have the same curvature at E; consequently, no parabola can pass between them, by art. 371. And these being contradictory, it follows, that, when the gravity in EH increases from E to H, the gravity at E in EH cannot be greater than the uniform gravity in the parabola ER. If the gravity in EH be supposed to decrease from E to H, let a parabola E*u* be described through any point *u* between R and Z, so as to have the same tangent and diameter at E with ER and EZ; then, since the gravity in EH at E is equal to the uniform gravity in the parabola EZ, (by the supposition) it must be greater than the uniform gravity in the parabola E*u* which is greater than the uniform gravity in ER. Therefore the part of the parabola E*u* that is adjoining to E passes between the curve EH and parabola ER; and (by art. 371.) ER has not the same curvature with EH, against the supposition. In like manner it may be shewn, that the gravity in EH at E is not less than the uniform gravity in the parabola ER. Therefore they are equal to each other. Let the common circle of curvature meet E*d* in B, and TR, ET and EB shall be in continued proportion, by art. 371. and TR is as the square of ET directly and EB inversely. Therefore the gravity in EH at

with which the body descends in the right line Dd, or by the second fluxion of DE the space described by it in this line, that is, by the second fluxion of the ordinate of the curve traced by the body on the immoveable plane $AGhs$, the time (or the right line AD which flows in the same manner as the time) being supposed to flow uniformly.

419. When a curve is described by a gravity that acts in parallel lines, its force is as the square of the velocity of the body directly and that chord of the circle of curvature which passes through the body in the direction of the gravity inversely. For, the same things being supposed as in the last article, let the velocity of the point D or the fluxion of AD be represented by a given line DG, let GH parallel to the ordinate DE meet the tangent ET in T; and the fluxion of the curve FE, or the velocity of the body in describing it, shall be measured by ET, by prop. 14. Let the circle of curvature at E meet Ed in B, and the second fluxion of DE shall be measured by a right line that is a third proportional to $\frac{1}{2}EB$ and ET, by prop. 33. Therefore the gravity, which is measured by the second fluxion of DE, is as the square of ET directly and EB inversely, that is, as the square of the velocity directly and the chord of the circle of curvature which passes through E in the direction of the gravity inversely.

420. This may be further illustrated, if it seem necessary, in the following manner. If the gravity act uniformly and in parallel lines upon a body that sets out from E in the direction ET, and bend its course into the curve ER, and the right lines TR, $V\varkappa$ in the direction of the gravity meet ET in T and V, and meet the curve in R and $\varkappa$, $V\varkappa$ shall be to TR as the square of EV is to the square of ET, by what was shewn in art. 95. consequently, the rectangle contained by $V\varkappa$ and an invariable right line is equal to the square of EV, and the curve $E\varkappa R$ is a parabola. And it may be shewn conversely, from art. 254. that when a parabola $E\varkappa R$ is described by a gravity that acts always in a direction parallel to the axis of the figure, then, DG being given, TR is invariable; and, consequently, the gravity is uniform. Let $HE\hbar$ be any curve line that touches the parabola $E\varkappa R$ and has the same curvature with it at E; then, if the power

by which EH is described act always in the same direction as the uniform gravity by which the parabola ER is described, and the velocities at E in EH and ER be equal, the gravity in the curve *b*EH at E must be equal to the uniform gravity in the parabola ER. For, if the gravity in EH at E be said to be greater than the uniform gravity in the parabola ER, let it be greater in the ratio of TZ to TR; and let EZ be a parabola described through E and Z that has the same tangent ET and diameter E*d* with the parabola ER at E. Then the gravity in the curve EH at E shall be equal to the uniform gravity in the parabola EZ; and, if the variable gravity in EH be supposed first to increase while the body moves from E to H, TZ shall be always less than TH, because a space described by a power that acts uniformly must be less than the space which is described in the same time when that power continually increases from the beginning of the time. But TR is always less than TZ; therefore the parabola EZ passes between the curve EH and parabola ER. But EH and ER were supposed to have the same curvature at E; consequently, no parabola can pass between them, by art. 371. And these being contradictory, it follows, that, when the gravity in EH increases from E to H, the gravity at E in EH cannot be greater than the uniform gravity in the parabola ER. If the gravity in EH be supposed to decrease from E to H, let a parabola E*u* be described through any point *u* between R and Z, so as to have the same tangent and diameter at E with ER and EZ; then, since the gravity in EH at E is equal to the uniform gravity in the parabola EZ, (by the supposition) it must be greater than the uniform gravity in the parabola E*u* which is greater than the uniform gravity in ER. Therefore the part of the parabola E*u* that is adjoining to E passes between the curve EH and parabola ER; and (by art. 371.) ER has not the same curvature with EH, against the supposition. In like manner it may be shewn, that the gravity in EH at E is not less than the uniform gravity in the parabola ER. Therefore they are equal to each other. Let the common circle of curvature meet E*d* in B, and TR, ET and EB shall be in continued proportion, by art. 371. and TR is as the square of ET directly and EB inversely. Therefore the gravity in EH at

at E is as the ſquare of the velocity at E (which is meaſured by ET, DG being given) directly and as the chord of the circle of curvature through E in the direction of the gravity inverſely; for the uniform gravity in the parabola ER is meaſured by 2TR, by art. 74. & 75.

421. When the velocity and direction of the motion and the force and direction of the gravity at E are given, the curvature at E is given. For, let ET be the ſpace that would be deſcribed in a given time by the motion at E continued uniformly, let TR be equal to the ſpace deſcribed in the ſame time by a body falling from E in the right line E*d* when the gravity at E is continued uniformly; then EB the chord of the circle of curvature ſhall be a third proportional to TR and ET; and, ſince the angle BET is given, the center and ray of curvature are given. The curvature therefore of EH at E depends only on the force and direction of the gravity at E, the velocity and direction of the projection at E being given, and not on the ſubſequent variations of the gravity. When the force and direction of the gravity with the velocity at E are given, EB is determined, and the ray of curvature is reciprocally as the ſine of the angle BET.

422. Let EH be now deſcribed by a centripetal force directed towards a given point S in the right line EB; and let ER be the parabola that would be deſcribed by the ſame centripetal force at E continued uniformly in a direction always parallel to EB: then the contact of the curve EH (that is deſcribed with a variable centripetal force directed towards S) with the parabola ER (that is deſcribed with the centripetal force in EH at E continued uniformly from that term) ſhall be cloſer than the contact of EH with a parabola that is deſcribed by a greater or leſs force continued uniformly in the ſame direction. Therefore the curve EH and parabola ER have the ſame curvature at E. From which it follows, that the centripetal force in the curve EH at E directed towards any point S is as the ſquare of the velocity of the motion at E directly and EB the chord of the circle of curvature that paſſes through S the center of the forces inverſely.

423. Let SE be produced till it meet the right line AD given in poſition; and, if AD be ſuppoſed to flow uniformly, the centripetal

Pl. XXI.
Pa. 350.

Fig. 181.

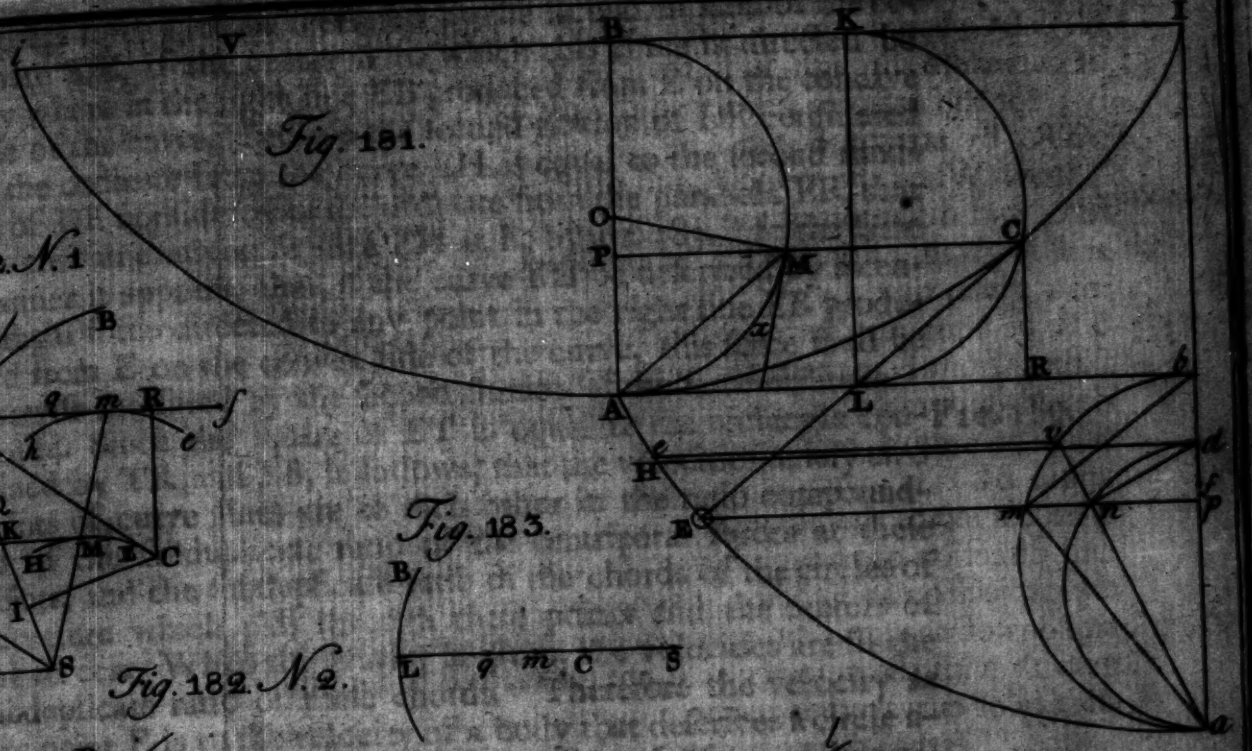


Fig. 182. N. 1

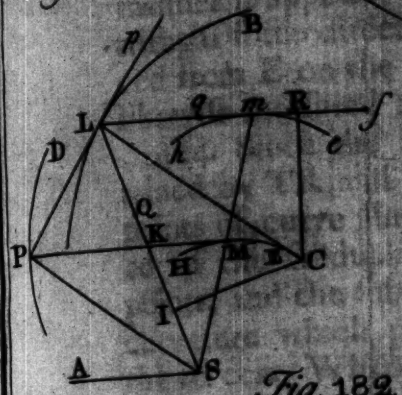


Fig. 183.

Fig. 182. N. 2.

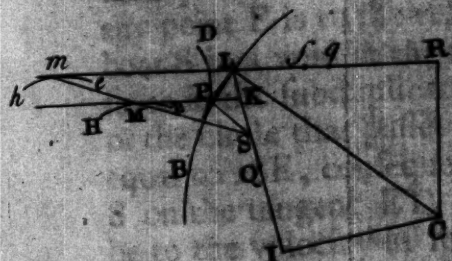


Fig. 184.

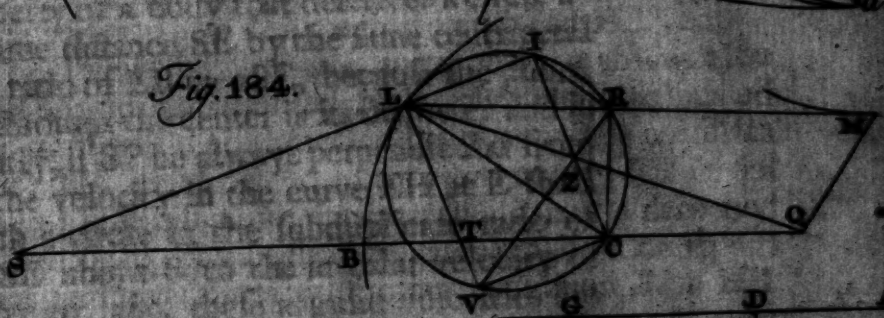


Fig. 185.

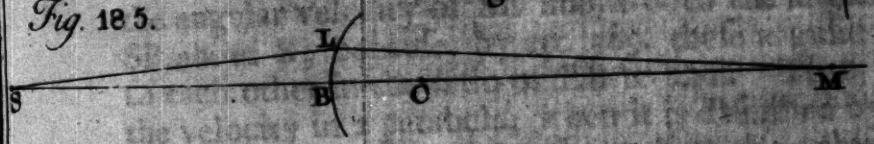


Fig. 186.

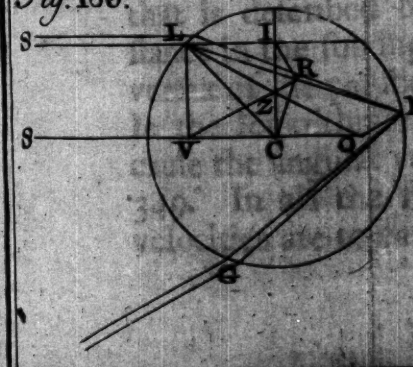


Fig. 187.

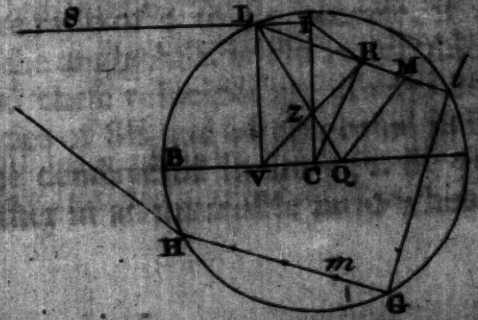
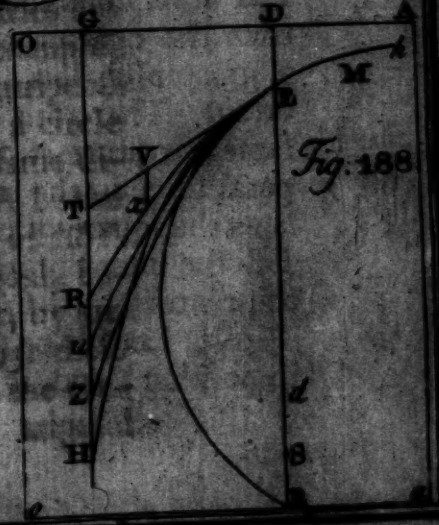
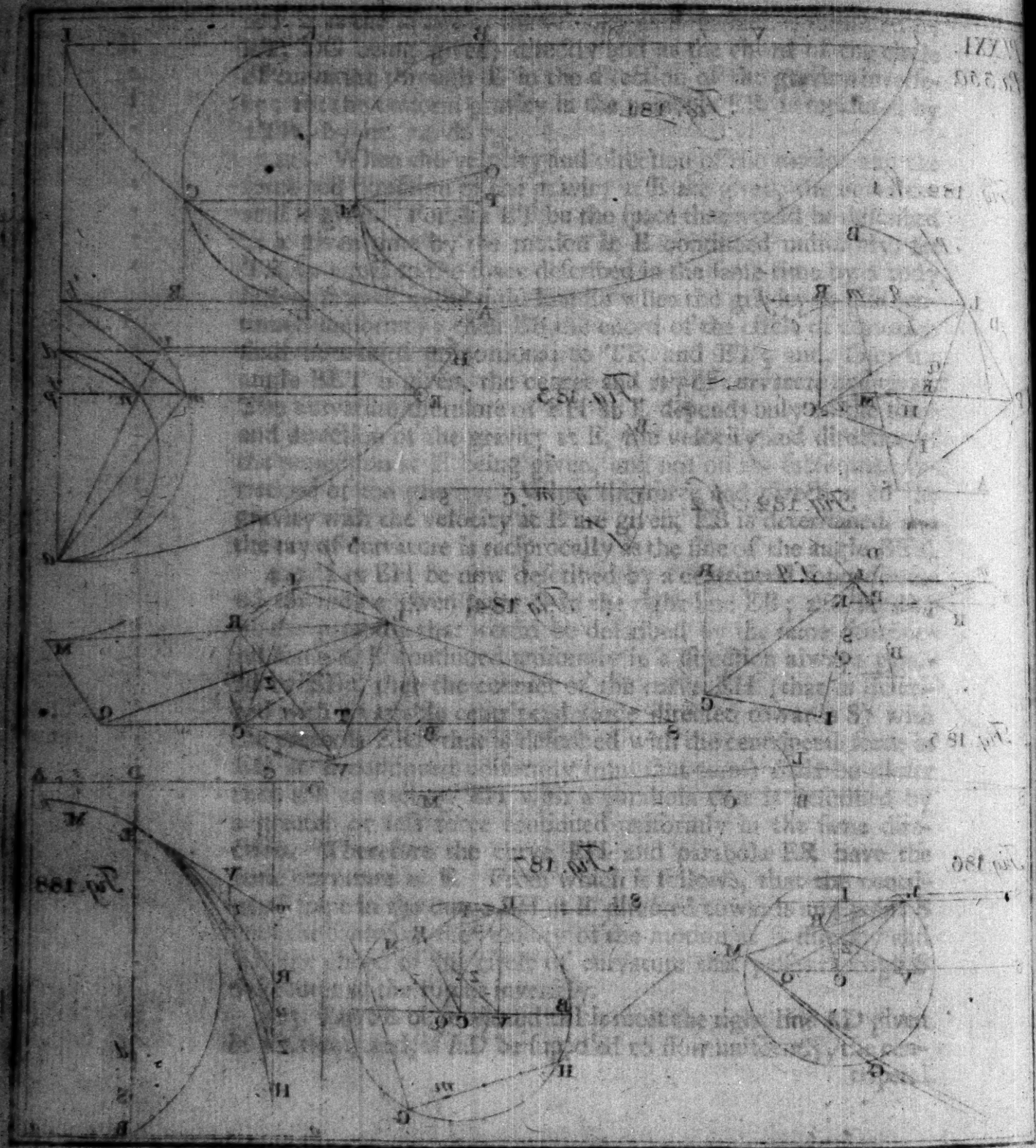


Fig. 188.





tripetal force ſhall be meaſured by the ſecond fluxion of the ordinate DE, if the center S to which this force is directed be any where in the right line EB produced from E on the concave ſide of the curve; becauſe the ſecond fluxion of DE conſidered as the ordinate from the curve EH is equal to the ſecond fluxion of DE conſidered as the ordinate from the parabola ER that has the ſame curvature with EH at E, by art. 383. In the ſame manner it appears, that, if the curve EH be deſcribed by a centrifugal force directed to any point in the right line EB produced from E on the convex ſide of the curve, this force ſhall be likewiſe meaſured by the ſecond fluxion of the ſame ordinate.

424. Since the ſquare of ET is equal to the rectangle contained by TR and EB, it follows, that the velocities at any two points of curve lines are to each other in the ratio compounded of the ſubduplicate ratio of the centripetal forces at theſe points, and the ſubduplicate ratio of the chords of the circles of curvature which paſs through theſe points and the centers of the forces. When the gravity is given, the velocities are in the ſubduplicate ratio of theſe chords. Therefore the velocity at any point E is to the velocity of a body that deſcribes a circle about S in a void at the ſame diſtance SE by the ſame centripetal force in the ſubduplicate ratio of EB to 2SE, becauſe the chord of the circle that paſſes through the center is its diameter and is equal to 2SE; conſequently, if SP be always perpendicular from S on the tangent ET, the velocity in the curve EH at E ſhall be to the velocity in ſuch a circle in the ſubduplicate ratio of the angular velocity of SE about S to the angular velocity of SP about S; becauſe, by art. 389. theſe angular velocities are to each other in the ratio of EB to 2SE. Thus, for example, the velocity in a parabola, when it is deſcribed by a centripetal force directed towards the focus, is to the velocity in a circle that is deſcribed by the ſame centripetal force at the ſame diſtance in the ſubduplicate ratio of 2 to 1; becauſe, if A be the vertex of the parabola, the angle ASE is always double of ASP. In the logarithmic ſpiral theſe velocities are always equal, becauſe the angular velocities of SE and SP are equal in it, by art. 349. In all the figures conſtructed in art. 392. & 393. theſe velocities are to each other in an invariable ratio when the centripetal FIG. 189.

Fig. 171. & 172. tripetal force is directed towards S, *viz.* in the ſubduplicate ratio of n to $n-1$ in thoſe of art. 392. and in the ſubduplicate ratio of n to $n+1$ in thoſe of art. 393. becauſe the angular velocities of SL and SP about S are to each other in the ſame invariable ratio as the angles ASL, ASP in thoſe figures.

Fig. 158. 425. When a conic ſection is deſcribed by a centripetal force directed towards any point S in the plane of the ſection, let O*a* the ſemidiameter parallel to the tangent at E meet SE in R; and the velocity in the ſection at E ſhall be to the velocity in a circle deſcribed in a void at the ſame diſtance SE by the ſame centripetal force as O*a* is to a mean proportional betwixt SE and RE. For, by the fourth property of the circle of curvature demonſtrated in art. 375. the rectangle RE*b* (E*b* being one half of EB) is equal to the ſquare of O*a*, and EB is to 2SE as the ſquare of O*a* is to the rectangle RES. When the center of the forces coincides with O the center of the figure, the velocity in the conic ſection is to the velocity in the circle as O*a* to OE; becauſe, in this caſe, SE and RE coincide with OE. When S is in the focus of the ſection, theſe velocities are to each other in the ratio of O*a* to a mean proportional between the diſtance SE and half the tranſverſe axis of the figure, (which in this caſe is equal to RE,) or in the ſubduplicate ratio of the diſtance of the revolving body from the other focus to half the tranſverſe axis; and theſe velocities are equal at the extremity of the ſhorter axis in the ellipſe.

426. The velocity and direction of the motion and the centripetal force at any given point E, with the center O of the conic ſection, towards which the force is ſuppoſed to be directed, being given, the ſection is determined that can be deſcribed by ſuch a centripetal force. For, if T*r* be equal to the ſpace that a body falling from E in the right line OE would deſcribe by its gravity at E continued uniformly in the ſame time that ET would be deſcribed by the motion in the curve at E continued uniformly, a third proportional to T*r* and ET ſhall be equal to the parameter of the diameter that paſſes through E; therefore the ſemidiameter O*a* parallel to ET is given in poſition and magnitude, and the conjugate ſemidiameter OE being likewiſe given, the conic ſection is determined.

427. When

427. When the conic ſection is deſcribed by a centripetal force that is directed towards the focus S, let T$\varkappa$ be the ſpace that would be deſcribed by a body falling from E in the right line ES with the gravity at E continued uniformly, in the ſame time that ET would be deſcribed by the motion of the body at E continued uniformly in the tangent; let EB be to ET as ET is to T$\varkappa$, and EQ be equal to one fourth part of EB: then a third proportional to SQ and SE ſhall be equal to the tranſverſe axis of the figure. Let SP be perpendicular from S on the tangent ET in P, biſect SE in r, join Pr, let the point O be taken upon the right line Pr (the ſame or the contrary way from P with r according as ES is greater or leſs than EQ) ſo that PO may be equal to one half of the tranſverſe axis, and O ſhall be the center of the figure, by what was ſhewn in art. 375. When EQ is equal to ES, the figure is a parabola, the vertex of which is determined in that article; when EQ is leſs than ES, the figure is an ellipſe; and when EQ is greater than ES, it is an hyperbola. It appears likewiſe, that, when the centripetal force and velocity at E with the diſtance SE are given, the tranſverſe axis is given, whatever the direction of the motion at E (or the angle SET) may be. FIG. 159.

428. Let EH be any curve deſcribed by a centripetal force directed towards S; let the force at E be repreſented by the right line EK, and KV be perpendicular to the tangent ET in V; then EV ſhall repreſent the force by which the velocity of a body is accelerated in a void if it is projected from E towards T, or retarded if it is projected towards t; and KV ſhall repreſent the force by which its courſe is inflected from the tangent ET, or Et. Becauſe KV is to EK as Eb is to the ray of curvature EC, and EK is as the ſquare of the velocity at E directly and Eb inverſely, it follows, that KV is as the ſquare of the velocity at E directly and the ray of curvature EC inverſely, and that the curvature of EH at E depends on the velocity at E and the force KV only. Hence, if the force EV be increaſed or diminiſhed by any new force that acts in the direction of the tangent, and is in any aſſignable ratio to the force EK, but the force KV remain, the curvature at E of the arch EH deſcribed by the body ſhall remain the ſame as if that new force had not acted. FIG. 189.

acted. If we ſuppoſe the ſame curve to be deſcribed by a centripetal force directed towards any other point *ſ*, and the velocity at E to be the ſame as formerly, let K*k* and S*l* parallel to the tangent E*k* meet *ſ*E in *k* and *l*, and C*d* be perpendicular to *ſ*E in *d*, then the centripetal force towards *ſ* ſhall be to the centripetal force that was directed towards S in the former caſe as E*b* is to E*d*, (art. 422.) that is, as E*k* is to EK, (becauſe the angle *d*E*k* is equal to E*bd*,) or as E*l* is to ES.

429. It follows from what is ſhewn in the laſt article, that, when a body deſcribes any curve EH by a centripetal force directed towards S, in a *medium* the reſiſtance of which is in an aſſignable ratio to the centripetal force, this force is ſtill at E as the ſquare of the velocity directly and the chord of the circle of curvature that paſſes through S inverſely. For the reſiſtance affects the force EV only which is in the direction of the tangent or of the motion of the body, and the velocity in the curve is accelerated or retarded in this caſe by the difference or ſum of the force EV and the reſiſtance; but the force KV that is perpendicular to the tangent (upon which the curvature of EH at E depends) remains the ſame as when the motion is in a void; and, ſince the curvature of EH at E is the ſame in the medium as in the void when the velocity and centripetal force at E are the ſame, it follows, that the centripetal force at E in the medium as well as in a void is as the ſquare of the velocity at E directly and E*b* inverſely. The velocity in the curve at E in the medium is to the velocity in a circle deſcribed in a void by the ſame centripetal force at the ſame diſtance, in the ſame ratio as if the curve was deſcribed in a void by a centripetal force directed to the ſame center S, that is, in the ſubduplicate ratio of the angular velocity of SE to the angular velocity of SP about S, by art. 424. Therefore what has been ſhewn in the preceeding articles on this ſubject is to be extended equally to both caſes; and for this reaſon we choſe firſt to ſhew how the velocity in the curve is compared with the velocity in a circle deſcribed by the ſame centripetal force at the ſame diſtance, before we ſhould enquire into the variations of the velocity and centripetal force in a given figure, which are different in a medium from what they are in a ſpace void of reſiſtance. Thus, if

if a logarithmic ſpiral, for example, be deſcribed either in a medium or in a void by a force directed to the center S, the velocity is always equal to the velocity in a circle deſcribed in a void at the ſame diſtance by the ſame centripetal force: And in any of the figures conſtructed in art. 392. & 393, theſe velocities are in an invariable ratio when S is the center of the forces.

430. It follows, that the centripetal forces at E directed towards a given point S by which the ſame curve EH can be deſcribed in a medium and in a void, are to each other in the duplicate ratio of the velocity at E in the medium to the velocity at the ſame point in a void; becauſe the chord of curvature is the ſame in both caſes.

431. Another theorem by which the centripetal forces may be diſcovered follows from what was ſhewn in art. 385. Let EV a circle deſcribed from the center S meet PM in V; and the ſecond fluxion of PM ſhall be equal to the difference of the ſecond fluxions of SM and PV when the points M and V ſet out together from E. Therefore, if we ſuppoſe that while the body M deſcribes the curve HEM by a force directed towards S, another body V revolves in the circle EV, the uniform angular velocity of SV about S being equal to the angular velocity of SM about S when M ſets out from E, and that a third body L deſcends or aſcends in the right line SE ſo that SL is always equal to SM; then the centripetal force of M in the curve HEM at E ſhall be equal to the difference of the centripetal force by which V is retained in the circle EV and of the force by which the motion of L is accelerated or retarded at E in the right line SE. For the centripetal force in the circle EV is meaſured by the ſecond fluxion of PV when V ſets out from E, AD being ſuppoſed to flow uniformly, by art. 423. and the force by which the motion of L is accelerated or retarded in the right line SE is meaſured by the ſecond fluxion of SL, or SM, when L comes to E, by art. 418. Therefore the difference of thoſe forces is meaſured by the difference of theſe ſecond fluxions of PV and SM, or the ſecond fluxion of PM, and, conſequently, is equal to the centripetal force in HEM at E, by art. 423. The centripetal force in the circle EV is e- FIG. 166.

qual to the centrifugal force that arises from the motion of rotation in the same circle, or in the curve EM at E, since (by the supposition) the angular velocities of SM and SV are equal when M and V set out from E. And hence it appears, that the force which accelerates or retards the velocity of L at E in the right line ES (which is sometimes called the paracentric velocity of M at E) is equal to the difference betwixt the centripetal force in EM at E and the centrifugal force in the circle EV, and not to the difference betwixt the centripetal force in EM at E and twice the centrifugal force in the circle EV. It follows from this, that, when two curve lines EM, *em* are described by centripetal forces directed towards S, and SM is always equal to S*m*, the difference of these forces at E and *e* must be equal to the difference of the centripetal forces in the circles EV, *ev*, the angular velocities of SV and S*v* being respectively equal to the angular velocities of SM and S*m* when M and *m* set out from E and *e*. And, if the velocity in the circle EV be to the velocity in the circle *ev* as G is to F, the difference of the forces in EM and *em* at E and *e* shall be to the force in the circle EV as the difference of the squares of G and F is to the square of G.

FIG. 190.

432. When a circle is described by a centripetal force that is directed towards its center, the motion is uniform; for, the direction of the centripetal force being always perpendicular to the tangent, it neither accelerates nor retards the velocity of the body, and has no other effect but to bend its course continually from the tangent into the circle. The centripetal forces are as the squares of the velocities directly and the diameters of the circles inversely, by art. 422. or (because the velocities are as the diameters directly and the times in which the revolutions are completed inversely) as the diameters directly and the squares of the periodic times inversely. If rays be supposed to be drawn always from the centers to the revolving bodies, the centripetal forces shall be likewise in the ratio compounded of the ratio of the angular velocities of those rays (or of the angular velocities of the tangents at the bodies) and the ratio of the velocities of the revolving bodies. Thus, when the circles are described in equal periodic times, or the angular velocities of

of thoſe rays are equal, the velocities and the centripetal forces are as the diſtances of the revolving bodies from the centers of the circles. When the ſquares of the periodic times are as the cubes of the rays of the circles, and, conſequently, the velocities in the ſubduplicate ratio of theſe rays inverſely, the centripetal forces are reciprocally as the ſquares of the rays. When the periodic times are as the ſquares of the rays, or the velocities reciprocally as the rays, the centripetal forces are reciprocally as the cubes of the rays. In general, when the velocities are reciprocally as any power of the rays whoſe exponent is m, the centripetal forces are reciprocally as the power of thoſe rays whoſe exponent is $2m+1$: And, converſely, when the centripetal forces are reciprocally as the power of the rays whoſe exponent is n, the velocities are reciprocally as the power of the rays whoſe exponent is one half of $m-1$.

433. The velocity in a circle is to the velocity that would be acquired by falling in a right line from the circumference to the center, by the ſame centripetal force with which the circle is deſcribed continued uniformly, as the radius is to the ſide of the inſcribed ſquare, or as unit is to the ſquare root of two. Let S be the center, SA the radius of the circle, and ſuppoſe the centripetal force at A to continue to act upon the body uniformly and in parallel lines; then ſhall it deſcribe a parabola AR that has the ſame curvature with the circle at A, or that has its focus in the point F, if SA be biſected in F, (by art. 422. & 371.) Let GA be equal to AF, and it is known that the velocity in this parabola at A is equal to the velocity that would be acquired by falling with the ſame uniform centripetal force from G to A; and therefore is to the velocity that would be acquired by falling from A to S with the ſame centripetal force in the ſubduplicate ratio of GA to AS, (art. 95.) or of 1 to 2; and, ſince the velocity in the circle is ſuppoſed equal to the velocity in the parabola in A, it muſt be to the velocity acquired by falling from A to S in the ſame ratio of 1 to the ſquare root of 2. Hence the periodic time in a circle is to the time in which a body would fall directly from the circumference to the center by the ſame gravity, as the circumference of the circle is to the ſide of the inſcribed ſquare. In general, it appears in the ſame manner, that

FIG. 191.

that the velocity in any curve at E is equal to the velocity that would be acquired in a void by falling directly through one fourth part of the chord of the circle of curvature EB, by the gravity at E continued uniformly; or that *Eb* is always double of the ſpace in falling through which the velocity at E would be generated in a void by the gravity at E continued uniformly.

FIG. 21. 434. It was ſhewn in art. 95. that, when a motion is uniformly accelerated, that is, when the force that accelerates it is conſtant, if *ad* be the ſpace deſcribed by the motion and the triangular area ADE be always equal to the rectangle *ae* contained by *ad* and the invariable right line *af*, the time of the motion ſhall be repreſented by the baſe AD, and the velocity by DE. The force that accelerates the motion is as IH the fluxion of the velocity directly and EI the fluxion of the time inverſely, (by art. 114. or 417.) and therefore may be repreſented by a right line *ab* that is to the invariable right line *af* as IH is to EI, or DE to AD, or as the ſquare of DE is to the rectangle ADE which is equal to twice the rectangle *fad*; conſequently, the ſquare of DE which meaſures the velocity at D is equal to twice the rectangle *bad* contained by *ba* which meaſures the centripetal force and *ad* the ſpace deſcribed from the beginning of the motion; and the fluxion of the ſquare of the velocity is meaſured by the double rectangle contained by *ba* and the right line which meaſures the fluxion of the ſpace.

FIG. 191. Hence, if a body be ſuppoſed to fall in the right line *a*S by a centripetal force in the direction *a*S that at any diſtance SQ is meaſured by the ordinate QN, the fluxion of the ſquare of the velocity at Q ſhall be meaſured by twice the rectangle contained by QN and the right line which meaſures the fluxion of *a*Q: but this rectangle meaſures the fluxion of the area *a*QN*d*, by prop. 3. therefore the ſquare of the velocity acquired by falling from *a* to Q is meaſured by twice the area *a*QN*d*, by art. 24. This, if it was neceſſary, might be demonſtrated in the ſame manner as the third propoſition, from this principle, which is analogous to the firſt axiom, That when a motion is accelerated by a centripetal force that increaſes continually, the velocity generated by it while a given ſpace is deſcribed, is greater than the velocity which would have been generated if the accelerating

ting force had not increased, but continued uniform. It is likewise obvious, that if aQ was a column of an homogeneous fluid, and the gravitation of any particle at the distance SQ was represented by the ordinate QN, the pressure of the fluid at Q would be as the area $aQNd$, or as the square of the velocity that would be acquired by falling from a to Q.

435. Let the velocity in the curve AE at A be equal to that which would be acquired by falling from a to A in the right line aS, so that its square may be represented by twice the area $aADd$; let SE be equal to SQ and the centripetal force at E be supposed equal to the centripetal force at Q which is represented by the ordinate QN; let EK be equal to QN, and KV be perpendicular to the tangent EP in V: then the force by which the motion in the curve at E is accelerated shall be represented by EV, and the fluxion of the square of the velocity at E by twice the rectangle contained by EV and the right line which measures the fluxion of the curve AE, by the last article: But this rectangle is equal to the rectangle contained by EK, or QN, and the right line which measures the fluxion of aQ, because (by prop. 17.) the fluxion of the curve AE is to the fluxion of aQ as EK is to EV; therefore the fluxion of the square of the velocity at E is measured by twice the fluxion of the area $aQNd$; and the square of the velocity at E is measured by $2aQNd$. The square of the same velocity is measured by twice the rectangle contained by QN and $\frac{1}{2}$EB, (by art. 433.) or by the rectangle contained by QN and Eb; consequently, the rectangle contained by QN and Eb is double of the area $aQNd$: and hence, when the centripetal force at E, or Q, is known, and the law according to which this force varies in different distances, Eb is found by the quadrature of the area $aQNd$. Let the square of the right line QR be equal to twice the area $aQNd$, that QR may represent the velocity at E; then QN, QR and Eb shall be in continued proportion; and, the fluxion of the square of QR being equal to twice the fluxion of the area $aQNd$, the fluxion of QR shall be to the fluxion of aQ as QN is to QR, or as QR is to Eb; that is, the fluxion of QR, which represents the velocity at E is to the fluxion of aQ, or $Sa - SE$, as QR is to Eb half of the chord EB.

436. Let

436. Let the ordinate AD repreſent the centripetal force at the diſtance SA, and the velocity in a circle deſcribed about the center S at the diſtance SA with the centripetal force AD ſhall be to the velocity acquired by falling from *a* to Q in the ſubduplicate ratio of the triangle SAD to the area *a*QN*d*, by art. 433. & 434. And when a body is projected upwards from A in the right line A*a* with a velocity equal to that in the circle, it will juſt riſe to the point *a*, and return again from thence, if the area AD*ad* be equal to the triangle SAD. When the centripetal force is reciprocally as any power of the diſtance whoſe exponent is any number *m* greater than unit, there is a limit which always exceeds the area AD*da* how great ſoever the height A*a* may be, by art. 293. & 325. and this limit is to the rectangle SAD (or twice the triangle SAD) as unit is to $m - 1$. Hence, if a body be projected upwards from A in the right line A*a* with a velocity that is to the velocity in a circle deſcribed at the diſtance SA with the centripetal force AD in the ſubduplicate ratio of 2 to $m - 1$, or in any greater ratio, the body will riſe for ever in the right line A*a*, and never return again to A. If the velocity of a body that moves in the right line S*a*, or in any curve AE, be at any point A to the velocity in a circle at the diſtance SA in the ſubduplicate ratio of 2 to $m - 1$, its velocity will be always to the velocity in a circle at the ſame diſtance from S the center of the forces in the ſame ratio, becauſe the ratio of the triangle SAD to the limit of the area AD*da* continued upwards from A is always the ſame where-ever the point A be taken in the right line S*a*. This velocity that is neceſſary to carry off a body for ever in the right line A*a* is greater or leſs than the velocity in a circle at the ſame diſtance according as 2 is greater or leſs than $m - 1$, that is, according as *m* is leſs or greater than 3; and theſe velocities are equal when *m* is equal to 3. When a body is projected with this velocity in any other direction than that of its gravity, E*b* is to SE as 2 is to $m - 1$, by art. 435. when *m* is leſs than 3, the body riſes for ever in one of thoſe parabolic figures that were conſtructed in art. 392. (Fig. 171.) when *m* is greater than 3, it approaches to the center till it fall into it in one of the figures that were conſtructed in art. 393. (Fig. 172.) Firſt, let

m

m be leſs than 3, let L be any point in the trajectory, LP the tangent, SP perpendicular to LP in P, and let n be to unit as 2 is to the exceſs of 3 above m; let the angle LSA be made on the ſame ſide of LS with LSP, and in the ſame ratio to LSP as n is to unit; let SA be to SL as the power of SP whoſe exponent is n is to the ſame power of SL: then ſhall A be the *Apſis* of the figure, or the point where the tangent becomes perpendicular to the ray drawn from the center of the forces to the point of contact. Let AM be a right line perpendicular to SA, and the trajectory AL may be conſtructed by this right line AM, as in art. 392. by drawing SM from S to any point in this right line, making always the angle ASL to ASM as n is to unit, and SL to SA as the power of SM whoſe exponent is n is to the ſame power of SA. The demonſtration is no more than the converſe of what was ſhewn in that article. For, ſuppoſing L to be any point in the trajectory, A the apſis where SL and SP coincide; and, ſince the velocity at L is always to the velocity in a circle deſcribed at the ſame diſtance by the ſame centripetal force in the ſubduplicate ratio of 2 to $m-1$, by what has been ſhewn, it follows, (by art. 424.) that the fluxion of the angle ASL is to the fluxion of the angle ASP, and the angle ASL to ASP (art. 33.) as 2 is to $m-1$; ſo that ASL muſt be to PSL as 2 is to $3-m$, or n to unit; conſequently, if we ſuppoſe AM to be perpendicular to SA, and the angle ASM to be to ASL as $3-m$ is to 2, the angles ASM, PSL ſhall be equal, and SP to SA as SL is to SM. Let the arches fL, kP deſcribed from the center S through L and P meet SA in f and k, and the fluxion of SL ſhall be to the fluxion of SP as the fluxion of the arch fL is to the fluxion of kP, (by art. 211. and 202.) that is in the ratio compounded of that of SL to SP and that of 2 to $m-1$: from which it follows, (by the converſe of art. 167.) that SP is to SA (or SL to SM) as the power of SL whoſe exponent is one half of $m-1$ is to the ſame power of SA, or that SL is to SA as the power of SM whoſe exponent is n is to the ſame power of SA; and that the trajectory is conſtructed in the manner we have deſcribed from art. 392. Becauſe the velocity of projection at A is greater than the velocity by which a circle is deſcribed at the ſame di- FIG. 171.

ſtance with the ſame centripetal force, the body begins to recede from the center at the apſis A, and it goes off in a parabolic curve by the conſtruction: When m is equal to 2, for example, the angle ASL is double of ASM, SA, SM and SL are in continued proportion, and AL is a common parabola that has its focus in S the center of the forces. Becauſe ASM never amounts to a right angle, ASL is to a right one in a ratio that is always leſs than that of 2 to $3 - m$, or that of n to unit: but if we ſhould ſuppoſe (as is uſual) the body to go off to infinity in the trajectory AL, the angle ASL deſcribed by the ray SL ſhall be to a right angle in that ratio; or, if the exceſs of 3 above m be expreſſed by the fraction $\frac{1}{s}$, the body will go off in a number of revolutions denoted by $\frac{1}{2}s$. Thus, if the centripetal force be reciprocally as the power of the diſtance whoſe exponent is $2\frac{99}{100}$, and the velocity of the projection at A be to the velocity in a circle deſcribed at the ſame diſtance with the ſame centripetal force in the ſubduplicate ratio of 200 to 199, the body will go off in 50 revolutions. If m be ſuppoſed ſucceſſively equal to $1\frac{1}{2}$, 2 and $2\frac{1}{2}$, and the velocity of the projection be to the velocity in a circle at the ſame diſtance in the ſubduplicate ratio of the number 4 to 1, 2 or 3, the body will go off in $\frac{1}{3}$, $\frac{1}{2}$, or one revolution, reſpectively. The body goes off in an hyperbolic curve when the ratio of thoſe velocities is greater than the ſubduplicate ratio of 2 to $m - 1$; and when the ratio of thoſe velocities is leſs, the body revolves continually about S within certain limits.

FIG. 172. 437. When m is greater than 3, and the velocity of projection at L is to the velocity in a circle deſcribed by the ſame centripetal force at the ſame diſtance in the ſubduplicate ratio of 2 to $m - 1$, let n be to unit as 2 is to the exceſs of m above 3, make the angle LSA in the ſame ratio to LSP as n is to unit, but on the oppoſite ſide of SL; let SA be to SL as the power of SL whoſe exponent is n is to the ſame power of SP; and A ſhall be the apſis of the trajectory. Upon the diameter SA deſcribe a ſemicircle SMA, and by it the curve AL is to be conſtructed (as in art. 393.) by drawing any right line as SM to the ſemicircle, making always the angle ASL to ASM as n is

is to unit, and SL to SA as the power of SM whoſe exponent is n is to the ſame power of SA: The demonſtration is ſimilar to that in the laſt article. When the centripetal force is reciprocally as the fourth power of the diſtance, m is equal to 4; and if the velocity of the projection be to the velocity in a circle at the ſame diſtance as the ſquare root of 2 is to the ſquare root of 3, then n is equal to 2, the angle ASL is double of ASM, the right lines SA, SM and SL are in continued proportion, and the trajectory AL is an epicycloid that is deſcribed by a point in the circumference of a circle while it revolves upon an equal circle, as we have ſhewn elſewhere. When m is equal to 5, 2 is to $m - 1$ as 1 to 2; and if the velocity of the projection be to the velocity in the circle as 1 to the ſquare root of 2, n is equal to unit, and the ſemicircle itſelf is the trajectory. When m is equal to 7, 2 is to $m - 1$ as 1 to 3; and if the velocity of the projection be to the velocity in the circle as 1 to the ſquare root of 3, n ſhall be equal to $\frac{1}{2}$, or ASL ſhall be one half of ASM, SL ſhall be a mean proportional betwixt SA and SM, and the trajectory is the ſame figure that is called the *lemniſcata* by the celebrated Mr. Bernoulli. It is manifeſt that the body falls always into the center S in a number of revolutions from the apſis A denoted by $\frac{1}{4}n$, or by $\frac{1}{2}s$ if the fraction $\frac{1}{s}$ be equal to the exceſs of m above 3. Thus, if m be equal to $3\frac{1}{100}$, and the velocity of projection at A be to the velocity in the circle at A in the ſubduplicate ratio of 200 to 201, the body will fall into the center after 50 revolutions; and the number of ſuch revolutions is inverſely as the exceſs of m above 3. The ſame conſtructions of the trajectories in this and the preceeding article were given long ago, *Deſcript. curv. par.* 2. *prop.* 22.

438. When m is equal to 3, and the velocity of the projection is equal to the velocity by which a circle is deſcribed at the ſame diſtance with the ſame centripetal force, the angular velocity of SE is equal to the angular velocity of SP; therefore the angles ESP, SPE are invariable, and the trajectory is the logarithmic ſpiral in which the body will approach to the center S, or recede from it, according as the direction of the Fig. 191.

projection forms an acute or an obtuſe angle with the ray drawn to the center S.

439. When m is equal to unit, or leſs, there is no aſſignable ſpace which the area AD*da* may not exceed by producing A*a* upwards, (by art. 325.) and a body projected upwards from A will return again how great ſoever the velocity of the projection may be. If we ſuppoſe a centrifugal force to be always directed from the point S that is inverſely as the power of the diſtance whoſe exponent is any number m leſs than unit, and a body to be projected at any point L with the velocity which it would acquire by this centrifugal force in moving from S to L, or that is to the velocity in a circle deſcribed at the ſame diſtance by a centripetal force equal to this centrifugal force in the ſubduplicate ratio of 2 to $1 - m$, the trajectory will be conſtructed by the right line AM in the ſame manner as in art. 436. only, in this caſe, n is a fraction that is to unit as 2 is to $3 - m$. Thus, if the centrifugal force be invariable, and the ratio of theſe velocities be that of the ſquare root of 2 to unit, the trajectory will be conſtructed by taking the angle ASL always equal to two thirds of the angle ASM and SL equal to the firſt of two mean proportionals betwixt SM and SA; for in this caſe n is to unit as 2 is to 3. If the centrifugal force be as the diſtance, n is equal to $\frac{1}{2}$, ASL is one half of ASM, SL is a mean proportional betwixt SM and SA, and the trajectory is an equilateral hyperbola that has its center in S. In all thoſe caſes the trajectory has an aſymptote that paſſes through S and conſtitutes with SA an angle which is to a right one as 2 is to the exceſs of 3 above m.

Fig. 192.

440. It may be of uſe, to avoid miſtakes that may ariſe in enquiries of this kind, and in other caſes where ſecond fluxions are introduced, to reflect here on what was ſhewn concerning theſe fluxions above, in art. 74. 97. and in ſeveral other places. The centripetal force in the curve AEM at E is moſt commonly meaſured by TM the ſubtenſe of the angle of contact at E when the tangent ET is ſuppoſed to be diminiſhed infinitely, the time in which the arch EM is deſcribed being given; that is, by Tr when ET which repreſents the velocity at E is of a finite magnitude (as we always ſuppoſe it) and

Fig. 191.

Er

Er is a parabola of the ſame curvature at E with AEM: becauſe *Tr* is the ſpace which the body would deſcribe by the gravity at E continued uniformly, in the ſame time that the body would deſcribe ET by its motion at E continued uniformly in the tangent. If therefore we ſuppoſe QN (which repreſents the centripetal force at the diſtance SQ, or SE) to be equal to *Tr*, the ſquare of the velocity at E will be repreſented by the rectangle contained by EB (or 2E*b*) and QN, becauſe the ſquare of ET is equal to the rectangle contained by EB and *Tr*. And, ſince the ſquare of the velocity that is acquired by falling through E*b* with the ſame gravity QN continued uniformly, is repreſented by twice the rectangle contained by E*b* and QN, (art. 434.) it ſhould follow, that the velocity in the curve at E is equal to the velocity that would be acquired in falling through E*b* by the gravity at E continued uniformly; whereas it has been ſhewn, (art. 433.) that theſe velocities are not equal, but are to each other in the ſubduplicate ratio of 1 to 2. In the ſame manner, the velocity in a circle at the diſtance SE would be found equal to the velocity that is acquired by falling through the radius ES with the gravity at E continued uniformly; whereas we found this velocity to be greater than the velocity in the circle in the ſubduplicate ratio of 2 to 1. The error ariſes from the ſuppoſing the gravity, when the body deſcends in the right line ES, or *a*Q, to be meaſured by the fluxion of the velocity of the body, or the ſecond fluxion of the ſpace deſcribed by it, and at the ſame time by the right line *Tr*: for theſe ſuppoſitions are inconſiſtent, becauſe it is 2*Tr*, and not *Tr*, that meaſures the ſecond fluxion of the ſpace deſcribed by the body while it deſcends in the right line *a*Q with a motion uniformly accelerated, as has been ſhewn by ſeveral different methods in art. 74. 75. 97. & 254. The gravity is meaſured by the increment of the velocity that is generated in a given time when the gravity is invariable (and, conſequently, the velocity increaſes uniformly,) or by the difference of the ſpaces that would be deſcribed in the given time by the velocities at the beginning and end of that time continued uniformly: but *Tr* is equal to the half of this difference only, being equal to the exceſs of the ſpace that is deſcribed in the ſame given time when the motion is uniform-

ly

ly accelerated above the ſpace that would have been deſcribed if the motion had continued uniform from the beginning of that time. Therefore, in order to avoid ſuppoſitions that are inconſiſtent, QN which meaſures the centripetal force at Q, or E, muſt be ſuppoſed equal to $2Tr$, the velocity at E in the curve is to be repreſented by a mean proportional betwixt $\frac{1}{2}$QN and EB, or QN and Eb, and the velocity in a circle at the diſtance SE by a mean proportional betwixt QN and the radius SE. The example we have deſcribed may ſerve to ſhew how miſtakes of this kind have ſometimes ariſen in the application of this method, and how they may be avoided.

441. When a circle is deſcribed by a centripetal force directed towards its center, the motion in the circle, the angular motion of the ray drawn from the center to the revolving body, the motion with which the area flows that is deſcribed by this ray, and the angular motion of the tangent at the body are all uniform. In other caſes, the area deſcribed by the ray SE drawn from the center of the forces S to the revolving body ſtill flows uniformly and meaſures the time; but the velocity at any point E is inverſely as SP the perpendicular from S on the tangent ET, the angular velocity of the ray SE about S is inverſely as the ſquare of SE, and the angular velocity of the tangent (or of the perpendicular SP about S) is inverſely as the rectangle contained by SP and the ray of curvature at E. Sir Isaac Newton's demonſtration of this may be repreſented in the following manner. When a body moves in a right line ET with an uniform motion, it deſcribes equal ſpaces on that line in any equal times; and if a ray be ſuppoſed to be drawn always from the body to a given point S that is not in that right line, this ray ſhall deſcribe equal areas about S in any equal times, Elem. 35. 1. A force that acts upon the body at any point E directed towards S has no effect on the magnitude of the area deſcribed in a given time by the right line drawn always from the body to S: it may accelerate or retard the motion of the body; but the area deſcribed about S is of the ſame magnitude as if no ſuch force had acted upon it. For, let ET be to EK as the velocity of the body in the direction ET is to the velocity that would be produced by the new impulſe

Fig. 193.

pulſe in the direction ES, complete the parallelogram EKMT, join ST and SM, the body will now deſcribe the diagonal EM in the ſame time that it would have deſcribed the ſide ET if no new force had acted upon it at E, and the triangle SEM is equal to SET. The ſame is to be ſaid of any ſucceſſive impulſes provided they be always directed to the ſame center S; they never affect the fluxion of the area, which therefore flows in the ſame manner as if the body had proceeded in the tangent with the motion at E continued uniformly. It follows from this, Fig. 191. that if AL be the tangent at A, and the triangle SAL be equal to the triangle SET, then AL and ET would be deſcribed in equal times by the motions at A and E continued uniformly. Therefore, if SX be perpendicular to AL in X, the velocity at A ſhall be to the velocity at E as AL is to ET, or as SP is to SX; that is, the velocity in any given figure is always inverſely as the perpendicular from the center of the forces on the tangent.

442. This may be likewiſe demonſtrated from art. 435. where it was ſhewn, that if the ſquare of QR be equal to twice the area *a*QN*d*, ſo that QR may repreſent the velocity at E, the fluxion of QR ſhall be to the fluxion of *a*Q as QR is to E*b*. The fluxion of SP is to the fluxion of SE, or SQ, as SP is to E*b*, by art. 384. and the velocity with which SQ decreaſes is equal to the velocity with which *a*Q increaſes; therefore the velocity with which SP decreaſes is to that with which QR increaſes as SP is to QR; and the fluxion of the rectangle contained by QR and SP vaniſhes, by prop. 3. conſequently, this rectangle is invariable, and QR which meaſures the velocity at E is inverſely as the perpendicular SP. Becauſe the area deſcribed about S flows uniformly and meaſures the time, the angular velocity of SE about S is reciprocally as the ſquare of the diſtance SE, by art. 120. and if the ſector ESI be equal to the Fig. 32. area ESH, the ſector ESI will be generated by SE if its angular motion about S be continued uniformly in the ſame time n. 2. that the area ESH is deſcribed by the ray drawn from S to the body while it moves in the arch EH. The angular velocity of the tangent at E, or of the perpendicular SP, is to the angular velocity of SE as SE is to E*b*, and is inverſely as Fig. 191. the

the rectangle SE*b*, or the rectangle contained by SP and the ray of curvature EC.

443. The centripetal force at any point E is as the ſquare of the velocity at E directly, and E*b* half the chord of the circle of curvature that paſſes through S inverſely, by art. 422. Therefore this force is inverſely as the ſolid contained by E*b* and the ſquare of the perpendicular SP, or (becauſe E*b* is to SP as the fluxion of the diſtance SE to the fluxion of the perpendicular SP, by art. 384.) inverſely as a ſolid that is to the cube of SP as the fluxion of SE is to the fluxion of SP. It is likewiſe as the angular velocity of SP (or of the tangent at E) directly and a third proportional to SE and SP inverſely. The ſame force may be meaſured by the difference of the ſecond fluxion of SE conſidered as terminated at the tangent E*t* and its ſecond fluxion conſidered as terminated at the curve EH, or by the ſum or difference of the centrifugal force that ariſes from the motion of rotation (which is always inverſely as the cube of the diſtance SE, by art. 442. ſince the angular velocity of SE is reciprocally as the ſquare of SE) and the force by which the motion of
Fig. 166. L is accelerated or retarded at E in the right line ES, as in art. 431. and by ſeveral other theorems of this kind.

Fig. 191. 444. When a figure has the ſame curvature and the tangents inclined in the ſame angle to the rays drawn to the center of the forces at any two points, the centripetal forces at theſe points are reciprocally as the ſquares of the diſtances. For, ſince the chord of the circle of curvature is the ſame and SP is to SE in the ſame ratio in both caſes, the ſolid contained by EB and the ſquare of SP is as the ſquare of SE. Thus, the centripetal forces are reciprocally as the ſquares of the diſtances at the extremities of the ſame axis in any conic ſection, when the center of the forces is any where in that axis, or in a circle at the extremities of the diameter that paſſes through the center of the forces.

Fig. 194. 445. Let AEM be an ellipſe, and the centripetal force be directed to the center of the figure; then, ſince EB is equal to the parameter of the diameter that paſſes through E, (by art. 374.) the rectangle contained by the ſemidiameter OE and E*b* is equal to the ſquare of the ſemiconjugate OK. The rect-

angle

angle contained by OK and OP the perpendicular from O on the tangent is invariable. Therefore OE is inverſely as the ſolid contained by the ſquare of OP and E*b*; and the centripetal force is as the diſtance OE. This is likewiſe eaſily deduced from what was ſhewn in the introduction, p. 8. That, when the area EOM deſcribed about O (and, conſequently, the time in which the arch EM is deſcribed) is given, then TM is in an invariable ratio to the diſtance OE. The velocity at E is as the ſemidiameter OK conjugate to OE; becauſe the perpendicular from O on the tangent at E is inverſely as OK. The periodic time in the ellipſe is equal to the periodic time in a circle deſcribed at any diſtance OE by the centripetal force at E: For, if the tangent E*t* be ſuppoſed equal to the circumference of a circle whoſe radius is OK, the triangle EO*t* ſhall be equal to the area of the ellipſe, and the periodic time in the ellipſe equal to the time in which E*t* would be deſcribed by the motion at E continued uniformly, by art. 441. but the velocity at E is to the velocity in the circle at the ſame diſtance as OK is to OE, (by art. 425.) or as E*t* is to the circumference of the circle; conſequently, the time in E*t*, or the periodic time in the ellipſe, is equal to the periodic time in the circle. It follows, that the periodic times in all ellipſes are equal (as well as in circles, art. 432.) when the centripetal forces are as the diſtances. Let OE be equal to OQ, and the velocity at any point E equal to the velocity that would be acquired by falling from *a* to Q; then ſhall O*a* be equal to AH the diſtance betwixt the extremities of the tranſverſe and ſhorter axis. For, if *ad* and QN repreſent the centripetal forces at *a* and Q, as in art. 435. the trapezium *a*QN*d* ſhall be to the triangle OQN in the duplicate ratio of the velocity at E in the ellipſe to the velocity in a circle at the ſame diſtance, or as the ſquare of OK to the ſquare of OE; and the ſquare of O*a* to the ſquare of OE as the ſum of the ſquares of OK and OE to the ſquare of OE; conſequently, the ſquare of O*a* is equal to the ſum of the ſquares of OK and OE, or to the ſquare of AH. Hence, if a body be projected at A in a direction perpendicular to OA with the velocity that is acquired by falling from *a* to A, and a circle deſcribed from the center A with a radius equal to O*a* meet

OH perpendicular to OA in H, then OA and OH ſhall be the two ſemiaxes of the trajectory. In the ſame manner it is ſhewn, that when a centrifugal force directed from O is as the diſtance, an hyperbola is deſcribed that has its center in O: and as the time of the motion is meaſured by an elliptical or circular area when the centripetal force is as the diſtance, ſo it is meaſured by an hyperbolic area when the centrifugal force is as the diſtance; that is, by the meaſures of angles in the former caſe, and by the meaſures of ratios in the latter; which is agreeable to what was ſhewn above in art. 159. & 407.

FIG. 159. 446. Let the center of the forces be now in the focus S of any conic ſection; let the circle of curvature at E meet ES in B, and BX parallel to the tangent at E meet EX perpendicular to the ſame tangent in X; let XZ be perpendicular to EB in Z, and XZ ſhall be equal to the parameter of the tranſverſe axis of the figure, by the 8th property of the circle of curvature, in art. 375. Becauſe the triangles SEP, BEX are ſimilar, the ſquare of SP is to the ſquare of SE as the ſquare of EX to the ſquare of EB, or as EZ is to EB; conſequently, the ſolid contained by EB and the ſquare of SP is equal to the ſolid contained by EZ and the ſquare of SE. Therefore, ſince EZ is invariable in a given figure, the centripetal force towards S is reciprocally as the ſquare of the diſtance SE. It follows from what was ſhewn above (art. 424. & 427.) that the velocity in the parabola is to the velocity in a circle at the ſame diſtance as the ſquare root of 2 to unit, that the velocity in the ellipſe is to the velocity in ſuch a circle in a leſs ratio, and that the velocity in the hyperbola is to that velocity in a

FIG. 195. greater ratio. The time in which a revolution is completed in an ellipſe, is equal to the time in which a circle is deſcribed by the ſame centripetal force at a diſtance equal to half the tranſverſe axis; for the velocity at H the extremity of the ſhorter axis is equal to the velocity in this circle, by art. 426. But if HT be ſuppoſed to be taken upon the tangent at H equal to the circumference of that circle, the triangle HST ſhall be equal to the area of the ellipſe, and the periodic time in the ellipſe equal to the time in which HT would be deſcribed by the motion at H continued uniformly in the tangent, (by art. 441.) and, conſequently,

ſequently, equal to the periodic time in the circle deſcribed at the diſtance SH. Hence, when ellipſes are deſcribed by centripetal forces that are always inverſely as the ſquares of the diſtances from their common focus, the ſquares of the periodic times are as the cubes of the tranſverſe axes or of the mean diſtances. And ſince the tranſverſe axis is determined when the velocity and centripetal force at any given point in the curve are given, (by art. 427.) it follows, that all bodies projected from a given point in different directions, but with equal velocities, complete their revolutions and return to the ſame point again in equal times. The tranſverſe axis was determined in art. 427. The ſhorter axes of ſuch ellipſes are to each other as the perpendiculars from the center of the forces on the right lines in which the bodies are projected, or as the ſines of the angles contained by theſe right lines and the ray drawn to the center of the forces; becauſe the ſquare of SP is to the ſquare of SE as EZ is to EB, and (ſince the tranſverſe axis with the diſtance SE and chord of curvature EB are given) the ſquare of SP is as the parameter EZ, or as the ſquare of the ſhorter axis. Let SE be equal to SQ, and the velocity in the ellipſe at E be equal to the velocity that would be acquired by falling from *a* to Q, and S*a* ſhall be equal to the tranſverſe axis of the figure. For, let S*h* be taken upon S*a* equal to SH, and *hl* be the ordinate of the figure *ad*NQ at *h*; then, becauſe the velocity at H in the ellipſe is equal to the velocity in a circle at the ſame diſtance, the triangle S*hl* ſhall be equal to the area *ahld*, (art. 436.) or to the difference of the rectangles S*hl* and S*ad*; conſequently, the triangle S*hl* is equal to the rectangle S*ad*, and 2S*a* is to S*h* as *hl* is to *ad*, that is, as the ſquare of S*a* is to the ſquare of S*h*; and S*a* is equal to 2S*h* or to the tranſverſe axis of the figure. Therefore, if a body was projected directly upwards from E in the right line SE with its motion at E, it would riſe to a diſtance from S equal to the tranſverſe axis of the figure; and the motion with which a circle is deſcribed would carry the body to a diſtance from S equal to the diameter of the circle. If a body begin to fall from *a* in the right line *a*S, it will be carried to an equal diſtance from S on the other ſide and will thence return again to *a* in the ſame time that a revolution is com-

FIG. 159.

FIG. 195.

completed in the ellipſe AEa whoſe tranſverſe axis is equal to Sa, and the time in which it falls from A to S is one fourth part of this time, or of the periodic time in the ellipſe AaA, or of the time in which a circle is deſcribed at a diſtance from S equal to one half of Sa. If a body be projected from A in a right line perpendicular to SA with the velocity that would be acquired by falling from a to A, then A ſhall be an apſis of the figure, and Sa the diſtance of the other apſis ſhall be equal to aA. When the velocity at E is to the velocity in a circle at the ſame diſtance as SE is to SP, the angular velocity of the ray SE is equal to the angular velocity of the ray in a circle deſcribed by the ſame centripetal force at the ſame diſtance: In this caſe, SE is equal to one half of the principal parameter of the figure, the tranſverſe axis is perpendicular to SE, the trajectory is an ellipſe, parabola or hyperbola, according as the angle SEP is greater, equal to, or leſs than half a right one; and the paracentric velocity in the trajectory is always greateſt at this point E. Becauſe the primary planets move in ellipſes that have their focus in the center of the ſun, ſo that the areas deſcribed by the ray drawn from any one planet to the ſun in equal times are equal, and the ſquares of the periodic times of the different planets are as the cubes of their mean diſtances from the ſun, (as was obſerved by the famous KEPLER,) and becauſe the ſame laws obtain in the motions of the ſatellites about the primary planets, Sir ISAAC NEWTON concludes, that there is a centripetal force extended over the ſyſtem to all diſtances directed towards the ſun that decreaſes in proportion as the ſquare of the diſtance from its center increaſes, and that a force decreaſing in the ſame manner is directed towards each body in the ſolar ſyſtem which at equal diſtances from their centers is as the matter in each of them. By comparing the effects of theſe forces with the deſcent of heavy bodies near the ſurface of the earth, he finds that the power of gravity ſo well known to us is one inſtance of this general principle. The powers upon which the celeſtial motions depend being thus diſcovered, and reduced to known meaſures, he then proceeds to deduce theſe motions with their various inequalities from their principles; and a fuller account of this matter is expected from an author who has already

dy given ſeveral proofs of the great progreſs he has made in this moſt uſeful theory.

447. Some find it difficult to conceive how a body can revolve in an ellipſe, and, after approaching towards the center of the forces in deſcending from the higher apſis A to the lower apſis a, recede from it again by aſcending from the lower apſis a to the higher A, ſince the centripetal force is greater at a than at A. There are indeed various laws of the centripetal forces, which would either cauſe the body to deſcend from the apſis continually towards the center and at length fall into it, or to aſcend from the apſis continually and recede from the center for ever: but there are other laws which if the centripetal force obſerve, the body may approach to the center and recede from it by turns; and of this kind is the law which obtains in the ſolar ſyſtem. In order to diſtinguiſh thoſe caſes from each other, we are to obſerve, that when the velocity of the body at the higher apſis A is leſs than that which is requiſite to carry it in a circle about the center S at the ſame diſtance SA, the body muſt move in a curve that falls within that circle there, and muſt approach towards the center; while it deſcends, its velocity increaſes; and if its velocity increaſe in a higher proportion than that in which the velocities requiſite to carry bodies in circles about S increaſe, the velocity in the lower part of the curve may at length exceed the velocity in a circle at the ſame diſtance, and thereby become ſufficient to carry off the body again. In theſe caſes the velocity of the body in the orbit and the velocity in a circle deſcribed at the ſame diſtance may exceed each other thus by turns, the latter in the higher part and the former in the lower part of the orbit, and the body may approach towards the center and recede from it by turns. Thus, in the ſolar ſyſtem, the centripetal force increaſes as the ſquare of the diſtance from the center decreaſes, and the velocity that is requiſite to carry a body in a circle about the center, increaſes only in the ſubduplicate ratio in which the diſtance decreaſes, art. 432, but the velocity in the orbit increaſes in a higher proportion while the diſtance decreaſes, (by art. 441.) ſo that though the former exceeds the latter at the higher apſis A, the latter by increaſing in a higher proportion becomes equal to it

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at the mean diſtance SH and exceeds it in its turn at the lower apſis a; and thus the body conſtantly revolves from one apſis to the other. This is further illuſtrated, by comparing the centripetal force in the orbit with the centrifugal force that ariſes from the circular motion of the body about S: for this centrifugal force increaſes always in proportion as the cube of the diſtance from the center decreaſes, and conſequently in a higher proportion than the centripetal force; ſo that each of theſe forces prevails in its turn, the centripetal force in the higher part, and the centrifugal force in the lower part of the orbit.

448. In general, a body may approach towards the center S and recede from it by turns when the velocities that are requiſite to carry bodies in circles about S increaſe in a leſs proportion than that in which the diſtances decreaſe. But if the centripetal force be ſuch, that theſe velocities increaſe in the ſame proportion in which the diſtances decreaſe, a body cannot revolve about the center S in this manner; if it begin to approach towards the center S when it proceeds from the apſis A, it will approach continually to the center till it fall into it; or if it begin to recede from S when it ſets out from the apſis a, it muſt recede from the center to greater and greater diſtances for ever. For, ſhould we ſuppoſe the body to deſcend in this caſe from the higher apſis A to the lower apſis a, ſince the velocity in the orbit at A would be to the velocity at a as Sa is to SA, (by art. 441.) that is, (by the ſuppoſition,) as the velocity in the circle at the diſtance SA is to the velocity in the circle at the diſtance Sa, it follows, that the velocity in the orbit at A would be to the velocity in a circle at the ſame diſtance as the velocity in the orbit at a is to the velocity in a circle at the diſtance Sa; and that, ſince the velocity in the orbit at A is leſs than the velocity in a circle at the diſtance SA, the velocity in the orbit at a would likewiſe be leſs than the velocity in a circle at the diſtance Sa; ſo that the body would continue to approach to the center after it comes to a. In the ſame manner, if the velocity at a in the orbit exceed the velocity in a circle at the diſtance Sa, the velocity in the orbit at A would likewiſe exceed the velocity in a circle at the diſtance SA, and the body would continue to recede from S after it comes to A.

When

When the centripetal force is reciprocally as the cube of the diſtance from S, the velocities that are requiſite to carry bodies in circles about S increaſe in the ſame proportion that the diſtances from S decreaſe, by art. 432. therefore the body after it proceeds from the apſis (unleſs it move in a circle) muſt approach continually to the center or recede from it for ever, and the figure can have no more than one apſis. When the centripetal force is reciprocally as any higher power of the diſtance, the velocities that are requiſite to carry bodies in circles about S increaſe in a higher proportion than that in which the diſtances decreaſe; conſequently, the body after it ſets out from the apſis either approaches continually to the center till it fall into it, or recedes from it for ever. Thus it was ſhewn in art. 440. that, when the centripetal force is reciprocally as a power of the diſtance whoſe exponent is any number that exceeds 3 by any fraction $\frac{1}{s}$, if it be projected at A with a velocity that is to the velocity in a circle at the diſtance SA in the ſubduplicate ratio of $2s$ to $2s+1$, it will fall into the center in a number of revolutions denoted by $\frac{1}{2}s$. But when the centripetal force is reciprocally as a power of the diſtance whoſe exponent is leſs than 3, the velocity in the trajectory increaſes while the diſtance decreaſes in a higher proportion than the velocity in circles deſcribed by the ſame centripetal forces, by art. 441. and in thoſe caſes the body may approach to the center and recede from it by turns.

449. In general, the velocity in any orbit becomes equal to the velocity in a circle deſcribed at the ſame diſtance by the ſame centripetal force when the angle ESP is a *maximum*, or when the angle contained by the tangent and ray drawn to the center of the forces is a *minimum*, by art. 424. or ſuppoſing (as in art. 435.) that the centripetal force at any diſtance SQ (or SE) is meaſured by the ordinate QN, and that the velocity at A is that which would be acquired by falling from a to A, when the area $aQNd$ becomes equal to the triangle SQN; that is, if the centripetal force be reciprocally as the power of the diſtance whoſe exponent is m, when the rectangle SQN is to the rectangle Sad as 2 is to $3-m$, or when the power of Sa whoſe expo- FIG. 191.

exponent is $m - 1$ is to the ſame power of SE in that ratio. Let a be the other apſis of the trajectory, and, SG being made equal to Sa, let GF repreſent the centripetal force at the diſtance SG, (or Sa,) and SA ſhall be to SG, in general, in the ſubduplicate ratio of the area $aGFd$ to the area $aADd$, (by art. 435. & 442.) that is, in the preſent ſuppoſition, in the ſubduplicate ratio of the difference betwixt the rectangles SGF and Sad to the difference of the rectangles SAD and Sad. The angular velocity of the ray SE in the trajectory becomes equal to the angular velocity in a circle deſcribed by the ſame centripetal force at the ſame diſtance, and the paracentric velocity is greateſt, in general, when the velocity in the curve is to the velocity in a circle at the ſame diſtance as SE is to SP, or when SE is to SA in the ſubduplicate ratio of the area $aADd$ to
Fig. 196. the triangle SQN. Thus, for example, if the centripetal force be the ſame in all diſtances, the velocity in the trajectory becomes equal to the velocity in a circle when SE is two thirds of Sa; and Sa (or SG) is to SA in the ſubduplicate ratio of aG to aA, from which it follows, that, if aA be biſected in K and a circle deſcribed through K from the center S meet AL perpendicular to SA in L, then Sa ſhall be equal to the ſum of KA and AL; and A is the higher or lower apſis according as SA is greater or leſs than this ſum; but when SA is equal to this ſum, a circle is deſcribed about the center S: When the cube of SE is double of the ſolid contained by aA and the ſquare of SA, the angular velocity of the ray SE is equal to the angular velocity of a ray in a circle that is deſcribed at the ſame diſtance by the ſame centripetal force, and the paracentric velocity is then greateſt at E.

Fig. 191. 450. In the logarithmic ſpiral, SP is to SE in an invariable ratio, Eb is equal to SE, and the centripetal force towards S is inverſely as the cube of SE, by art. 443. In the reciprocal or hyperbolic ſpiral (deſcribed in art. 344.) ST is invariable; from which it follows, that the fluxion of SP is to the fluxion of SE as the cube of ST to the cube of ET, or the cube of SP to the cube of SE, and that SP is to Eb in the ſame ratio; conſequently, the centripetal force towards S is inverſely as the cube of the diſtance SE. The centripetal forces obſerve the

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ſame law when the ſquare of SE is to the ſquare of SP as the ſum or difference of any given ſpace and the ſquare of SE is to a given ſquare. The conſtruction of the figures which have this property is given, *Harmon. menſur. p.* 31. and *Philoſ. Tranſ. n.* 317. Fig. 171. When any figure conſtructed in art. 392. is deſcribed by a centripetal force directed towards S, this force is inverſely as the power of the diſtance SL whoſe exponent is $3 - \frac{2}{n}$, becauſe LI is to SL as n is to $n - 1$, SP to SL as SA is to SM, and, conſequently, SP always as the power of SL whoſe exponent is $1 - \frac{1}{n}$. When any of the figures conſtructed in art. Fig. 172. 393. are deſcribed by a force directed towards S, SP is as the power of SL whoſe exponent is $1 + \frac{1}{n}$, LI is to SL in an invariable ratio, and the force is reciprocally as the power of SL whoſe exponent is $3 + \frac{2}{n}$: And this is the converſe of what was ſhewn above in art. 436. & 437.

451. Let AM be any curve line that can be deſcribed by a Fig. 173. force that is as the power of the diſtance whoſe exponent is any number m, let the angle ASL be to ASM as $m + 3$ is to 2, and SL be to SA as the power of SM whoſe exponent is one half of $m + 3$ is to the ſame power of SA; then the curve AL may be deſcribed by a centripetal force directed towards S, that is, as the power of the diſtance SL whoſe exponent is $\frac{4}{m+3} - 3$. The demonſtration may be deduced from art. 394, but will appear more eaſily afterwards. Thus, if m be ſuppoſed ſucceſſively equal to 2, 1, $\frac{1}{2}$, 0, $-\frac{1}{2}$, $-\frac{2}{3}$, the curve AL may be deſcribed by a centripetal force directed towards S that is inverſely as the power of the diſtance whoſe exponent is $2\frac{1}{7}$, 2, $1\frac{6}{7}$, $1\frac{4}{5}$, $1\frac{2}{3}$, $1\frac{1}{3}$, reſpectively. The point A is an apſis common to AM and AL. If B be the other apſis of AM and D the other apſis of AL, the angle ASD muſt be to ASB as $m + 3$ is to 2, by the conſtruction: Therefore, if $M - 3$ be ſuppoſed equal to m, and $N - 3$ equal to $\frac{4}{m+3} - 3$, the angle ASD

 ſhall

ſhall be to ASB in the ſubduplicate ratio of M to N. And this is agreeable to what is ſhewn by Sir ISAAC NEWTON of the motion of the apſides (*Princip. lib.* 1. *prop.* 45. *ex.* 2.) when the excentricity of the orbit is ſuppoſed incomparably little, which caſe only he has conſidered.

FIG. 196. 452. Suppoſe now that the revolving body deſcribes any trajectory AEM in a medium that reſiſts its motion. Let the centripetal force at E by which the curve would be deſcribed in a void be to the centripetal force at E by which it is deſcribed in the medium as any given invariable right line S*a* is to SZ, then the reſiſtance at E ſhall be inverſely as a ſpace that is to the ſquare of SP the perpendicular from S on the tangent as the fluxion of the curve AE is to the fluxion of SZ; and the denſity of the medium (ſuppoſing the reſiſtance to be as the denſity of the medium and ſquare of the velocity together) ſhall be reciprocally as a right line that is to SZ in the ſame ratio. For, let the velocity at E in the void be repreſented by E*l* and the centripetal force by E*k*, the velocity at E in the medium by EL and the centripetal force by EK, the reſiſtance by ER, and, the fluxion of the curve being repreſented by EN, let NV be perpendicular to the tangent in V that EV may repreſent the fluxion of the ray SE. When the direction of the motion at E is in the right line ET that forms an acute angle with the ray SE, the velocity is accelerated by a force that is to the centripetal force EK as EV is to EN the fluxion of the curve, and is at the ſame time retarded by the reſiſtance ER; the rectangle contained by 2EN and the difference of theſe two forces (that is, the exceſs of the rectangle 2KEV above the rectangle 2NER) meaſures the fluxion of the ſquare of EL, by art. 424. and the fluxion of the ſolid contained by S*a* and the ſquare of EL is meaſured by the ſolid contained by S*a* and 2KEV — 2NER. But the ſquare of EL is to the ſquare of E*l* as EK is to E*k* (by art. 429.) or as SZ is to S*a*; and the ſolid contained by S*a* and the ſquare of EL is always equal to the ſolid contained by SZ and the ſquare of E*l*; conſequently, their fluxions are equal, and the ſolid contained by S*a* and 2KEV — 2NER is equal to the fluxion of the ſolid contained by SZ and the ſquare of E*l*; but this fluxion conſiſts of two parts, the ſolid contained

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ed by SZ and the ſpace which meaſures the fluxion of the ſquare of *El*, (that is, the ſolid contained by SZ, 2E*k* and EV, by art. 435. or the ſolid contained by S*a*, 2EK and EV, ſince SZ is to S*a* as EK is to E*k*,) and the ſolid contained by the ſquare of *El* and the fluxion of SZ. From which it appears, that this laſt ſolid is equal to the ſolid contained by S*a*, 2NE and ER; but that the fluxion of SZ is negative, or that SZ muſt decreaſe while the body approaches to S in the arch EM and the ray SE decreaſes. Therefore, if *Zz* meaſure the fluxion of SZ, the rectangle contained by 2S*a* and EN ſhall be to the ſquare of *El* as *Zz* is to ER that meaſures the reſiſtance at E; and, becauſe *El* that meaſures the velocity in a void at E is always inverſely as SP, it follows, that the reſiſtance at E is directly as *Zz* the right line which meaſures the fluxion of SZ and inverſely as the ſolid contained by the ſquare of SP and EN the right line that meaſures the fluxion of the curve AE. When the direction of the motion at E is in the right line E*t* that forms an obtuſe angle with the ray SE, the velocity is retarded by a force that is to the centripetal force EK as EV is to EN, and at the ſame time by the reſiſtance ER; and it will appear in the ſame manner, that the reſiſtance ER is directly as *Zz* and inverſely as the ſolid contained by the ſquare of SP and EN, but that in this caſe SZ muſt decreaſe while the ray SE increaſes and the body recedes from the center. Nor can a body move in the trajectory, either by proceeding from E towards M or towards A, unleſs SZ begin to decreaſe from the moment when the body ſets out from E. If the area ASE be ſuppoſed to flow uniformly, the fluxion of the curve AE ſhall coincide with the velocity by which it is deſcribed in a void (by art. 441.) ſo that we may ſuppoſe *El* equal to EN; and 2S*a* ſhall be to EN as *Zz* is to ER, or the reſiſtance ſhall be always as the rectangle contained by the right lines that meaſure the fluxions of the curve AE and of SZ; and this coincides with prop. 23. par. 2. *Deſcript. curv.*

453. Becauſe ER is to *Zz* as the ſquare of *El* is to the rectangle contained by 2S*a* and EN, and the ſquare of EL is to the ſquare of *El* as SZ is to S*a*, ER is to *Zz* as the ſquare of EL to the rectangle contained by SZ and EN. Therefore the

denſity (which is as ER directly and the ſquare of EL inverſely) is as Zz directly and the rectangle contained by SZ and EN inverſely; conſequently, if the curve AEM be extended into a right line and the ordinate at E be always equal to SZ, the denſity at E ſhall be always inverſely as the ſubtangent of this figure.

454. Let AEM be any trajectory that can be deſcribed in a void by a centripetal force that is inverſely as any power of the diſtance SE, and let it be deſcribed in a medium by a centripetal force that is likewiſe inverſely as ſome power of the diſtance SE; let ST perpendicular to SE meet the tangent in T, and the denſity at E ſhall be always inverſely as the tangent ET. For let Ek and EK be inverſely as the powers of the diſtance whoſe exponents are n and m reſpectively; then SZ ſhall be always as the power of the diſtance whoſe exponent is $n - m$, and Zz the fluxion of SZ ſhall be to the fluxion of SE in the ratio compounded of that of SZ to SE and that of $n - m$ to unit, by art. 167. therefore the denſity ſhall be inverſely as a right line that is to SE as the fluxion of the curve AE is to the fluxion of the ray SE, that is inverſely as the tangent ET.

455. The reſiſtance at E is to the centripetal force in the medium at E as the rectangle contained by one half of Eb and Zz is to the rectangle contained by EN and SZ, or (becauſe Eb is to SP as the fluxion of SE to the fluxion of SP, by art. 384.) in the ratio compounded of the ratios of SP to 2SZ, of PE to SE and of Zz to the fluxion of SP. For example, if the trajectory be the logarithmic ſpiral, and EK the centripetal force in the medium be inverſely as the power of the diſtance SE whoſe exponent is m; then, ſince Ek is inverſely as the cube of the diſtance SE, SZ muſt be as the power of SP whoſe exponent is $3 - m$, the fluxion of SZ muſt be to the fluxion of SP in the ratio compounded of that of SZ to SP and that of $3 - m$ to unit, by art. 167. and the reſiſtance to the centripetal force in the ratio compounded of that of $\frac{1}{2}$PE to SE and that of $3 - m$ to unit. The body cannot deſcend towards the center S in this ſpiral unleſs 3 be greater than m; and if it aſcend in the ſpiral, m muſt be greater than 3, becauſe SZ muſt decreaſe in both caſes,

caſes, and becauſe SE is to SP in an invariable ratio. The denſity of the medium is inverſely as ET or the diſtance from S; as is ſhewn by Sir ISAAC NEWTON, *Princip. lib.* 2. *prop.* 15. and by Mr. BERNOULLI, *Mem. de l' Acad. Royale des Sciences* 1711. If we ſuppoſe the trajectory to be one of thoſe conſtructed in art. 392. or 393, and r be ſuppoſed equal to $3 - \frac{2}{n}$ in the former, or to $3 + \frac{2}{n}$ in the latter, SZ will be as the power of the diſtance whoſe exponent is $r - m$ (by art. 452.) and SP as the power of the diſtance whoſe exponent is one half of $r - 1$. Therefore (by art. 167.) the reſiſtance at any point L ſhall be to the centripetal force at L in the ratio compounded of that of LP to SL and that of $r - m$ to $r - 1$. The body cannot approach to S in any of thoſe figures unleſs r be greater than m, and it cannot recede from S in any of them unleſs r be leſs than m. The reſiſtance and denſity vaniſh at the apſis A in theſe figures, or in any of thoſe that can be deſcribed in a void by a force that is as any power of the diſtance. It was ſhewn above, (art. 429.) that the ratio of the velocity in the curve to the velocity in a circle deſcribed in a void at the ſame diſtance by the ſame centripetal force is always the ſame as when the curve is deſcribed in a void. FIG. 170. & 171.

456. If the centripetal force act in parallel lines, let AD perpendicular to thoſe lines flow uniformly; and let DZ be now to a given right line D*a* as EK, which meaſures the centripetal force in the medium at E, is to E*k* that meaſures the centripetal force in the void at E or the ſecond fluxion of the ordinate DE (by art. 418.) and, the reſt remaining, the reſiſtance ER will be to Z*z* as EN is to 2D*a*. If the centripetal force in the medium be ſuppoſed uniform, the rectangle contained by DZ and E*k* muſt be invariable, and the fluxion of DZ to the fluxion of E*k* (or the third fluxion of the ordinate DE) as DZ is to E*k*, (by prop. 3.) or as the rectangle contained by EK and D*a* is to the ſquare of E*k*. Therefore the reſiſtance ER is to the gravity EK as the rectangle contained by the right lines that meaſure the fluxion of the curve AE and the third fluxion of the ordinate DE is to twice the ſquare of E*k* that meaſures FIG. 197.

the

the ſecond fluxion of the ſame ordinate. The velocity at E in the medium is to the velocity at E in the void in the ſubduplicate ratio of EK to E*k*; and the denſity of the medium at E is reciprocally as a right line that is to E*k* as the fluxion of the curve is to the fluxion of E*k*, or as the right line that meaſures the third fluxion of the ordinate DE directly and the rectangle contained by the right lines that meaſure the fluxion of the curve and the ſecond fluxion of DE inverſely. By theſe theorems the reſiſtance and denſity of the medium may be computed when the nature of the curve is known; but they may be repreſented geometrically in the following general manner.

FIG. 149. & 150. 457. Suppoſe, as in art. 366. that the rectangle MTK is always equal to the ſquare of ET, that BV the tangent of the curve BKF at B meets ET the tangent of EM in V; then, if the curve be deſcribed by an uniform gravity that acts always in lines parallel to EB, in a medium whoſe reſiſtance is as its denſity and the ſquare of the velocity of the body together, the reſiſtance at E ſhall be to the gravity as 3EB is to 4EV, and the denſity of the medium ſhall be always inverſely as the tangent EV. But if the reſiſtance be ſuppoſed to be as the denſity and velocity together, let the angle EV*u* be made equal to EBV on the ſame ſide of EV, and V*u* meet EB in *u*; then the denſity ſhall be inverſely as $\sqrt{Eu}$. Let the figure, for example, be any conic ſection, O the center, EG a chord in the direction of the gravity; let the angle EO*k* be made equal to GET and O*k* meet EG in *k*, and the reſiſtance ſhall be to the gravity as 3O*k* is to 2OE; and if the tangent at E meet the ſemidiameter that biſects EG in V, the denſity ſhall be inverſely as EV: for, ſince the triangles EV*b*, OE*k* are ſimilar (by the 5th property of the circle of curvature, art. 375.) E*b* is to EV as O*k* to OE. When the ſection is a circle, O*k* becomes perpendicular to E*b*. When the ſection is an hyperbola and EB is parallel to one of the aſymptotes, the tangents BV and ET interſect each other in that aſymptote at V, and the reſiſtance is to the gravity as 3EV is to 2AV. When the tangent in any figure becomes perpendicular to EB the direction in which the gravity acts, the ratio of the reſiſtance to half the gravity is the ſame ratio which Sir ISAAC NEWTON calls the index of the variation.

FIG. 158.

FIG. 154.

of

of curvature, or the ratio of the fluxion of the ray of curvature to the fluxion of the curve. In computing the reſiſtance of the medium from the ſecond and third fluxions of the ordinate, or the right lines that repreſent them, regard muſt be had to what was ſhewn of theſe fluxions in chap. 4. and art. 255. in order to avoid miſtakes like to thoſe deſcribed in art. 440.

458. When the gravity acts in parallel lines, and is either uniform or varies as any power of the diſtance from a given plane GH, and AEH is any trajectory that could be deſcribed in a void by a force that is alſo as any power of the diſtance from GH, the denſity of the medium at E is always inverſely as the tangent ET terminated by the point of contact E and by GH in T. The demonſtration is ſimilar to that of art. 454. Fig. 198.

459. Let AEH be any trajectory deſcribed in a void, or in a medium that has no reſiſtance, by a force that acts always in right lines (as EM) perpendicular to GH and in the plane of the trajectory. Upon EQ that is parallel to GH take QN ſo as always to repreſent the force at the diſtance CQ (or EM) from GH; let AD repreſent the force at the diſtance CA, and the velocity at A be ſuch as would be acquired by falling from *a* to A with an uniform gravity equal to AD; complete the rectangle AD*da*; and the velocity at E ſhall be to the velocity at A in the ſubduplicate ratio of the area *a*QND*d* to the rectangle *a*D. Let AK and EP parallel to GH and equal to each other repreſent the conſtant fluxion of QE; let KI and PR meet the tangents AI and ER in I and R; let *ab* be to *a*A as the ſquare of AK to the ſquare of AI, and *bg* parallel to AD meet D*d* in *g*: then ER ſhall be to AI as the velocity at E to the velocity at A, or in the ſubduplicate ratio of *a*QN*d* to *a*D; conſequently, PR is to EP, or the fluxion of the ordinate EM to the fluxion of the baſe CM, in the ſubduplicate ratio of the area *b*QND*g* to the rectangle *ag*; and hence the conſtruction of theſe trajectories may be deduced when the law of the force is given. For example, when the force is inverſely as the cube of the diſtance from GH, the trajectory is a conic ſection. When the force is as the diſtance from GH, let A be the apſis of the trajectory; take QE always in the ſame ratio to the arch AL whoſe coſine is CQ (CA being radius) as *a*A is to one Fig. 199.

one half of CA; then E ſhall be in the trajectory, and the body will move from A to the right line CH always in the
FIG. 198. ſame time. When the force is inverſely as the ſquare of the diſtance from GH, if C*b* be equal to 2CA, the trajectory is a ſemicubical parabola that has its cuſpis at H ſo that CH is to a mean proportional betwixt CA and *ab* as 2 is to 3; and the cube of EM is equal to the ſolid contained by the ſquare of HM and a third proportional to *ab* and $\frac{1}{2}$ CA. It is conſtructed by the area of the circle when C*b* is leſs than 2CA, and by an hyperbolic area when C*b* is greater than 2CA. In general, QE is always as the time in which a body would deſcribe AQ by falling from A to Q by the ſame forces directed towards C.

FIG. 198. 460. The ſine of the angle MER is to the ſine of CAI in the ſame ratio, when the velocity at A with the diſtance AQ of the parallels AK and QE is given, and the force perpendicular to GH or QE is the ſame at the ſame diſtance from QE. For AI is to AK as the radius to the ſine of the angle AIK, or CAI, and ER is to EP as the radius to the ſine of the angle MER; conſequently, the ſine of the angle MER is to the ſine of CAI as AI to ER, or in the ſubduplicate ratio of *a*D to the area *a*QN*d*; but this ratio remains the ſame in all the different poſitions of the tangent AI: therefore, when a ray of light paſſes through any given medium that acts upon it in parallel lines perpendicular to the planes that terminate the medium, the ſine of the angle of refraction is to the ſine of the angle of incidence in an invariable ratio, and the velocity with which the ray emerges at E is to the velocity of its incidence at A in the inverſe ratio; as Sir ISAAC NEWTON has ſhewn in a different man-
FIG. 200. ner, *prop.* 94 & 95. *lib.* 1. *Princip.* If we ſuppoſe the ray to move from E towards A, and the angle MER to be increaſed till its ſine be to the radius in the ſubduplicate ratio of *a*D to *a*QN*d*, the angle CAI will become a right one, or A will be the apſis of the trajectory, and the ray will be reflected from A ſo as, after returning to the right line EQ*e* at *e*, to emerge in an angle equal to the angle of incidence MER.

461. When the body is projected from A in a right line AI that is not in the plane ACH, the force being ſtill perpendicular

Fig. 189.

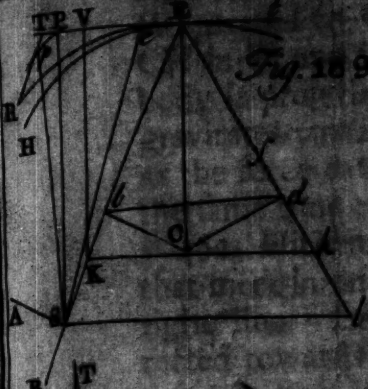


Fig. 190.

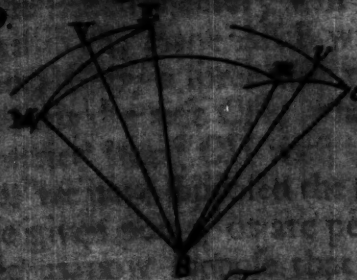


Fig. 191. N. 1.

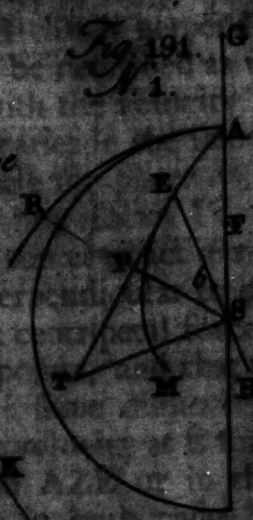


Fig. 191. N. 2.

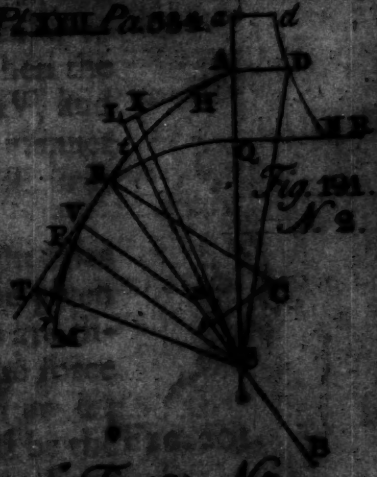


Fig. 193.

Fig. 194.

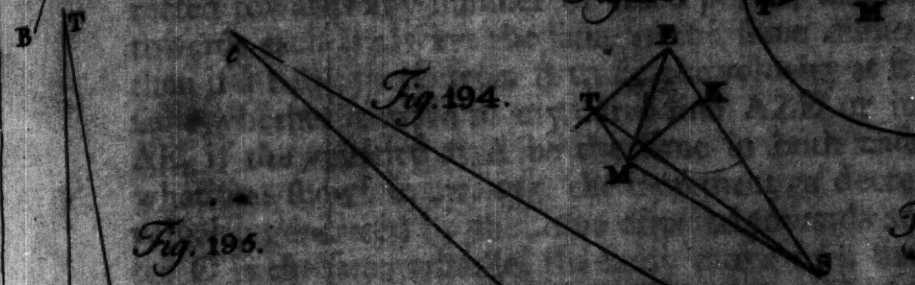


Fig. 191. N. 3.

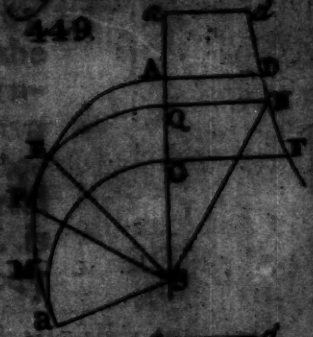


Fig. 195.

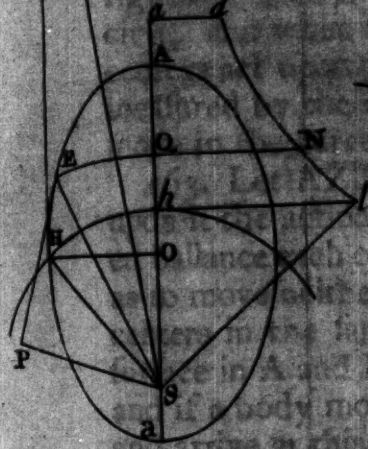
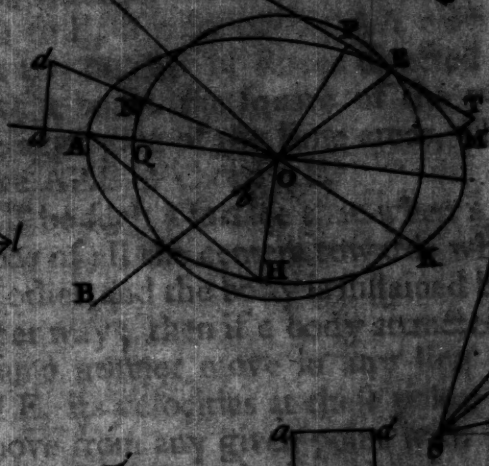


Fig. 192.



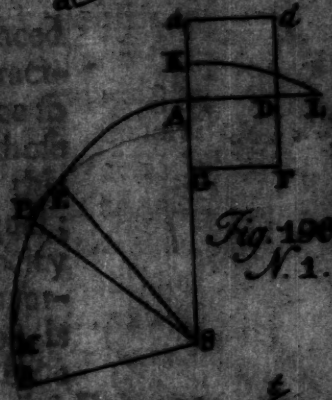


Fig. 196. N. 1.

Fig. 198.

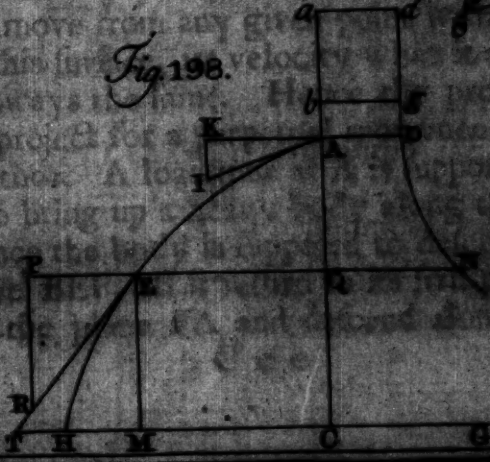


Fig. 197.

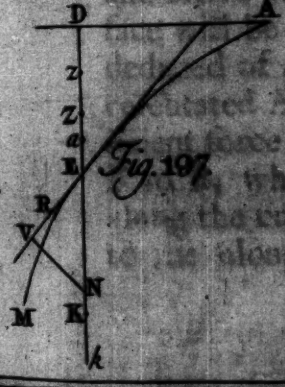
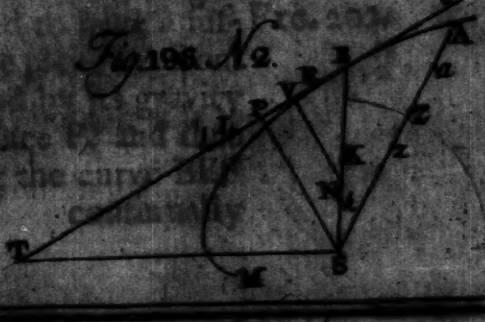
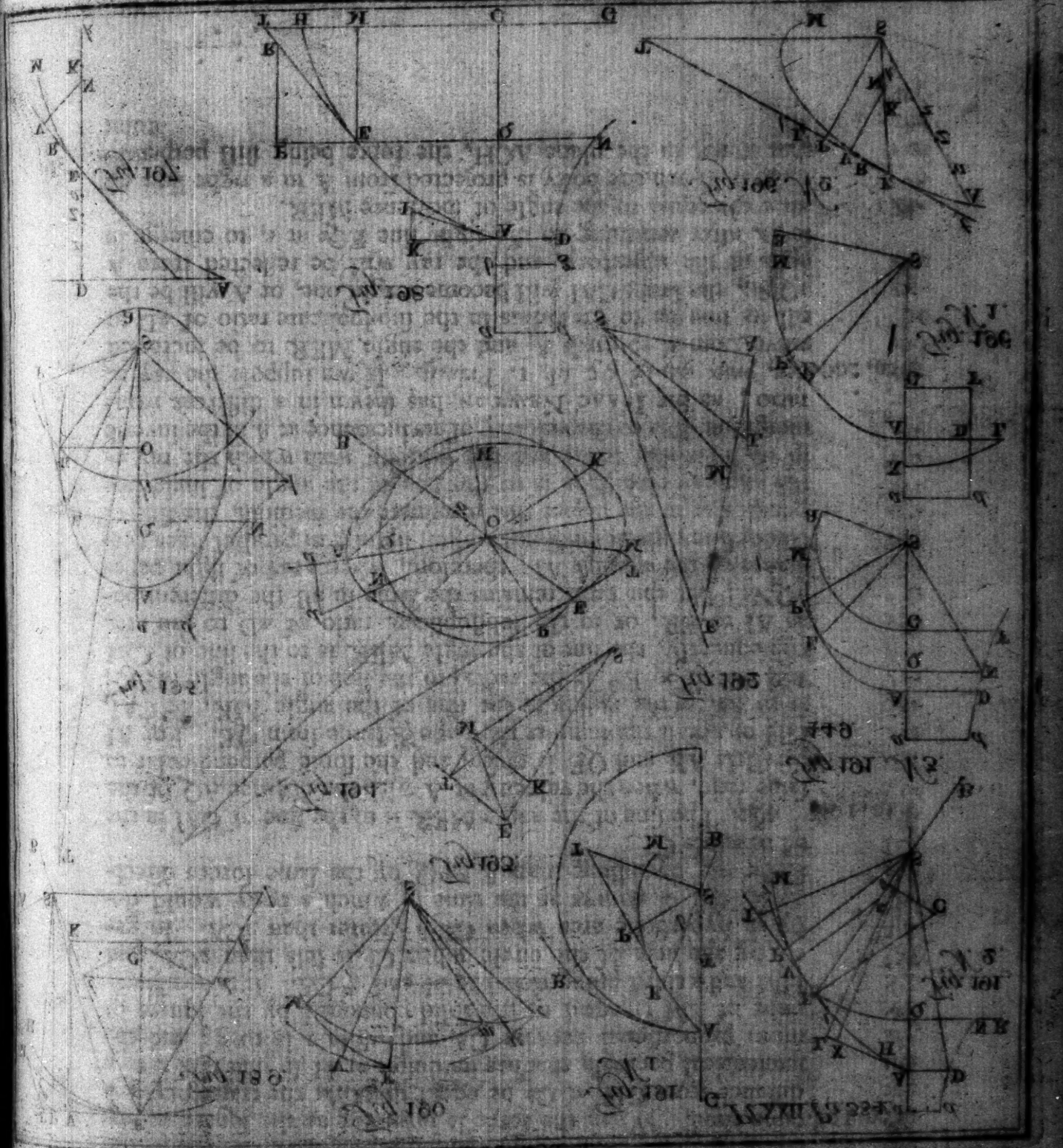


Fig. 196. N. 2.





cular to CH, let AI repreſent the velocity of the projection, and IV parallel to CH meet a plane AKB that is perpendicular to CH in V; and the ſame trajectory will be deſcribed as when the body is projected in the plane AKB with the velocity AV, and gravitates towards C by a force that varies in the ſame manner as the force towards the axis CH, while the plane AKBC is at the ſame time carried along CH with the velocity IV.

462. Hitherto we have ſuppoſed the forces to act in right lines that meet in one given point, or are perpendicular to one given right line. Let us now ſuppoſe that centripetal forces are directed towards any number of fixed points, and that the force towards each is always the ſame at the ſame diſtance from it; then if a body move from A to E, the velocity at E ſhall be the ſame whether it move in any curve line AZE or in the chord AE, if the velocity at A be the ſame in both caſes. For by what was ſhewn in art. 435. the increment or decrement of the velocity generated by the force directed towards any one center C is the ſame whether the body move in a curve or in a right line from A to E, if it ſet out from A with the ſame velocity; and when any number of forces directed towards ſeveral centers act upon the body, the ſquare of the velocity at E is meaſured by the aggregate of the ſame areas, whether the body move in the curve AZE or chord AE. FIG. 201.

463. Let AZE be ſuch a ſurface that when a body is placed upon it the actions of all the centers towards which it is attracted ballance each other, and the body is ſuſtained by the ſurface ſo as to move neither way; then if a body attracted towards thoſe centers in the ſame manner move in any line that meets this ſurface in A and E, its velocities at theſe points ſhall be equal; and if a body move from any given point with a given velocity and arrive at this ſurface, its velocity when it comes to this ſurface will be always the ſame. Hence the proper reſolution is deduced of a project for a perpetual movement mentioned by a celebrated Author. A loadſtone at A is ſuppoſed to have a ſufficient force to bring up a heavy body along the plane FA from F to B, whence the body is ſuppoſed to deſcend by its gravity along the curve BEF till it return to its firſt place F, and thus to riſe along the plane FA and deſcend along the curve BEF FIG. 202.

continually. But ſuppoſing BZE to be the ſurface upon which if a body was placed the attraction of the loadſtone and the gravity of the body would ballance each other, this ſurface ſhall meet BEF at ſome point E betwixt A and F, and the body muſt ſtop in deſcending along AEF at the point E. See WILKINS *Mathem. mag. Book* II. *Chap.* 13.

FIG. 201. 464. Let AZE be a trajectory deſcribed by forces directed towards the centers C and S, the ſquare of the velocity at A be repreſented by the rectangle *a*D, and the forces at A towards C and S by the ordinates AD and A*d* perpendicular to CA and SA, reſpectively; let CQ and S*q* be taken upon CA and SA reſpectively equal to CE and SE, and the ordinates QN, *qn* always repreſent the forces towards C and S at the Diſtances CE and SE; then if the ſquare of V be equal to twice the aggregate of the areas *a*D, AQND and A*qnd* joined with their proper ſigns, V ſhall meaſure the velocity at E. And in the ſame manner V is determined when there are more centers that attract the body. Let EH be taken upon EC, and EL upon ES reſpectively equal to QN and *qn*, complete the parallelogram EHKL, then the force at E that reſults from the attraction of the ſeveral centers ſhall act in the direction EK and be meaſured by EK. Let the circle of curvature at E meet EK in B, biſect EB in *b*, and the rectangle *b*EK ſhall be equal to the ſquare of V V, by art. 440. And if KR parallel to the tangent at E meet any other chord of the circle of curvature, as ED, in R, and ED be biſected in *d*, the rectangle *d*ER ſhall be equal to the ſquare of V; for the angle ERK is equal to EBD, EB to ED as ER to EK, and the rectangle *d*ER equal to *b*EK. Hence, if KP be perpendicular to the tangent at E in P, the rectangle contained by KP and the ray of curvature is equal to the ſquare of V; and if KR meet TE (that is drawn from E to any fixed point T) in *x*, the rectangle contained by E*x* and half the chord of curvature that paſſes through T ſhall be equal to the ſquare of V.

FIG. 203. 465. Let T be any fixed point, TC the baſe of the figure, and TA perpendicular to TC; let EM the ordinate from E meet the baſe in M and EN parallel to the baſe meet TA in N; let KH parallel to TA meet EM in H; then the time being always

ways ſuppoſed to flow uniformly, and the fluxion of the curve being repreſented by V which meaſures the velocity at E, the ſecond fluxion of the ordinate ſhall be always meaſured by KH, and the ſecond fluxion of the baſe by EH, or KL: That is, if while E deſcribes the trajectory, EM and EN be always perpendicular to TC and TA in M and N, the powers by which the motions of the points M and N are accelerated or retarded ſhall be meaſured by EL and EH. For let E*t* be taken upon the tangent equal to V, and *t*I parallel to TA meet EN in I, that E*t*, EI and I*t* may meaſure the fluxions of the curve baſe and ordinate, reſpectively. Let *tt* meaſure the ſecond fluxion of the ordinate, and ſuppoſing O to be the center of curvature, let Om be perpendicular to EM in m; and KP being perpendicular to E*t*, let P*z* be perpendicular to EN in *z*. Then the ſecond fluxion of the curve (or the power that accelerates the motion of E) ſhall be meaſured by EP. The rectangle contained by OE and I*t* is equal to that which is contained by Om and E*t*, and (becauſe the fluxions of thoſe rectangles are equal, and while CE increaſes Om and I*t* decreaſe) the rectangle contained by EI and E*t* is equal to the ſum of the two rectangles contained by Om and EP, and by OE and *tt*. By the laſt article KP is to E*t* (or V) as E*t* is to OE; and the ſum of KH and P*z* is to KP as EI to E*t*; therefore the Sum of the two rectangles contained by OE and KH, and by OE and P*z*, is equal to the rectangle contained by EI and E*t*, or to the ſum of the two rectangles contained by OE and *tt* and by Om and EP; but (becauſe of the ſimilar triangles OmE and P*z*E) the rectangle contained by OE and P*z* is equal to that which is contained by Om and EP; conſequently, the rectangle contained by OE and KH is equal to the rectangle contained by OE and *tt*, and KH is equal to *tt* which was ſuppoſed to meaſure the ſecond fluxion of the ordinate EM. In the ſame manner it appears that KL, or EH, meaſures the ſecond fluxion of the baſe TM. This theorem holds, not only when the angle MTN is right, but when MTN, TME and TNE are any given angles, providing KL parallel to TC meet EM in L, and KH parallel to TA meet EN in H; for KH will always meaſure the ſecond fluxion of TN, and KL the ſecond fluxion of TM.

466. This principle however, that when the force which results from the attraction of all the centers is measured by EK, and is resolved into the forces EH and EL parallel to TA and TC respectively, the second fluxions of the ordinate EM and base TM are measured by EH and EL respectively, may be admitted without this Proof. For if we resolve the motion at E in the direction E*t* into a motion EI in a right line parallel to TC, and a motion E*i* parallel to TA, we may conceive the body to descend in the right line EN with a velocity EI that is accelerated by the force EH, while the right line EN moves parallel to itself along TA with a velocity E*i* that is retarded by the force EL; and thus it may appear (as in art. 418.) that the force EH measures the second fluxion of EN, and the force EL the second fluxion of EM. The position of EH and EL serve likewise to shew whether these second fluxions are to be considered as affirmative or negative: thus when TM decreases and EM increases, and EK is within the angle MEN, the second fluxions of TM and EM are to be both considered as negative; but if EK be within the angle *i*EN, the second fluxion of EM is to be considered as positive.

467. This principle being admitted, several of the theorems in the preceeding articles may be readily demonstrated from it. Thus if the force EK be directed towards any fixed point that is any where in the right line EM, EK will coincide with EL, so that EH will vanish; consequently the centripetal force will be measured by the second fluxion of the ordinate EM when TM flows uniformly, as in art. 423. When the force is directed towards the fixed point T, the area described about T flows uniformly, or the fluxion of the area CTE is invariable; for the area CTE is half the sum of the areas CME and CTNE, the fluxions of which are measured by the rectangles IM and *i*N, by prop. 4. The fluxion of the rectangle IM is measured by the sum of the rectangles HM and I*i*, and the fluxion of *i*N by the sum of the rectangles LN and I*i*, by prop. 3. But while CM and ME increase, EN and I*t* decrease; consequently the second fluxion of the area CTE (or the fluxion of the sum of the rectangles IM and *i*N) is measured by the excess of HM and I*i* above LN and I*i*, that is, by the excess of HM above LN, or of KM above KN.

But

But when EK paſſes through T, this exceſs vaniſhes, and the area deſcribed about T flows uniformly; which is the firſt propoſition of the firſt book of Sir ISAAC NEWTON's principles, and was demonſtrated after his method above in art. 441. and in a manner that differs little from Mr. HERMAN's in art. 442.

468. In general it appears that the ſecond fluxion of the area CTE is meaſured by the exceſs of the rectangle KM above KN, or (if EK meet TA in V) by the rectangle contained by KL and TV; and the fluxion of the area CTE increaſes or decreaſes according as EK is on the ſide of ET towards which the body moves, or on the oppoſite ſide. The theorem in art. 431. may be likewiſe made more general; for let *Eu* be a circle deſcribed about the center T, and the uniform angular velocity of T*u* be equal to the angular velocity of TE while E moves in the trajectory. Let *up* parallel to TE meet the trajectory in *x*, and meet T*p* perpendicular to TE in *p*; and let *e* move in the right line TE ſo that T*e* be always equal to T*x*: then (by art. 385.) the ſecond fluxion of T*e* (or T*x*) ſhall be equal to the difference of the ſecond fluxions of *px* and *pu* when *x* and *u* ſet out together from E. But if K*y* be perpendicular to TE in *y*, E*y* ſhall meaſure the ſecond fluxion of *px* at that term; and the force by which the circle E*u* is deſcribed about T, is meaſured by the ſecond fluxion of *pu* at the ſame term, by art. 422. From which it follows, that the force by which the motion of *e* is accelerated or retarded, when it ſets out from E, is equal to the difference of the force E*y* and the force by which the circle E*u* is deſcribed about T, or the centrifugal force that ariſes from the circulatory motion of E about T. Of this principle, ſee *the Laws of the Moon's motion according to gravity*, p. 65 & 66. FIG. 203. n. 2.

469. We ſuppoſed the trajectory deſcribed by E to be in one plane in art. 466 & 467. But ſimilar concluſions are eaſily deduced when the trajectory *c*D*e* is not in one plane. In this caſe let CE be the orthographic projection of the trajectory on the plane CTA, or *e*E be always perpendicular from the trajectory to CTA in E, and EK be the projection of the right line *ek* which repreſents the force that reſults from the action of the ſeveral centers upon the body at *e* with its direction. Then the ſecond fluxions of EM and EN ſhall be meaſured by EL and EH; FIG. 204.

EH; and the ſecond fluxion of *c*E, the diſtance of the body from the plane CTA ſhall be meaſured by the difference of the perpendiculars *k*K and *c*E. From which it follows, that when a trajectory is deſcribed by a force that is directed towards a fixed point T, and another force that is always perpendicular to the plane CTA, the area deſcribed by TE about T on this plane flows uniformly.

FIG. 204. 470. Let a body ſet out from a given point D with a given velocity, and firſt let the trajectory be deſcribed in the plane CTA. Let the force at D that reſults from the attraction of the ſeveral centers be repreſented by D*f*, and be reſolved into D*l* and D*b* parallel to TA and TC, reſpectively. Let DF and DG be perpendicular to TC and TA in F and G; upon which produced take F*f* equal to D*l*, and G*g* equal to D*b*. Let M*m* and N*n* be taken in the ſame manner upon EM and EN always equal to EH and EL, reſpectively. Let the velocity at D in the direction DG parallel to CT be ſuch as would be acquired by a body falling from B to F with an uniform gravity equal to the force F*f*; and let the velocity at D in the direction parallel to TA be ſuch as would be acquired in the ſame manner by a body falling from A to G with an uniform gravity equal to the force G*g*; or, in general, let theſe velocities be ſuch as that half the ſquare of each may be meaſured by the rectangles F*b* and G*a*, according to art. 434. Suppoſe likewiſe that while the body moves from D to E, TM decreaſes, but that EM increaſes, and that EK falls within the angle MEN. Then the ſquare of the velocity of M (or of the fluxion of the baſe TM) ſhall be meaſured by 2BM*mfb*, and the ſquare of the velocity of N (or of the fluxion of the ordinate EM) by 2G*a* — 2GN*ng*. For let M*q* meaſure the velocity of M, or the fluxion of CM, and the fluxion of this velocity (or ſecond fluxion of CM) being meaſured by EH or M*m* (by art. 465.) it follows that the fluxion of the ſquare of M*q* is equal to the fluxion of 2BM*mfb*, and that the ſquare of M*q* is equal to 2BM*mfb*. In the ſame manner the ſquare of the velocity of N is meaſured by 2G*a* — 2GN*ng*. And the fluxion of the baſe TM is to the fluxion of the ordinate EM in the ſubduplicate ratio of the area BM*mfb* to G*a* — GN*ng*. When the trajectory *c*D*e* is not in the plane CTA, the motions

at

at D and E are each to be reſolved into three motions in the directions parallel to CT, TA, and that which is perpendicular to CTA; the forces at D and E are each to be reſolved likewiſe into three forces in theſe directions; and the fluxions of TM, ME and Ee are to be determined in the ſame manner as the fluxions of TM and ME in the former caſe.

471. There ariſes hence a ſimple enough conſtruction of the trajectory that would be deſcribed by the moon about the earth, if, the gravity towards the ſun being inverſely as the ſquare of the diſtance, the gravity towards the earth varied in the ſame proportion as the diſtance from its center, and we ſhould abſtract from the curvature of the earth's orbit during a revolution of the moon; which we ſhall ſubjoin, becauſe it ſerves to illuſtrate ſome parts of the theory of the moon's motion. For though the gravity towards the earth be not as the diſtance from its center, but inverſely as the ſquare of that diſtance; yet when the orbit of the moon is ſuppoſed to approach nearly to a circle, ſome of the effects of the ſolar action deduced from theſe ſuppoſitions will nearly coincide, and the trajectory is more eaſily determined in the former caſe than in the latter. Let S repreſent the Fig. 205. ſun, T the earth, CADB the orbit of the moon about the earth, C and D the quadratures, A the conjunction, B the oppoſition, E the moon's place in this orbit at any time, and EN a perpendicular from E on ST that joins the centers of the ſun and earth. If the ſun acted on the earth and moon with equal forces, and in parallel lines, they would fall equally towards the ſun in parallel lines, and the ſolar action would not diſturb the motion of the moon and earth about their common center of gravity, or the motion of the moon about the earth, when the motion is all referred to the moon, as is uſual. But ſince the ſun acts with greater force upon the moon at A than upon the earth at T, and upon the earth at T with more force than upon the moon at B; it follows that, if the earth and moon fell towards the ſun, the moon would fall more than the earth towards the ſun in the former caſe, and the earth more than the moon in the latter; ſo that their diſtance from each other would increaſe by the inequality of the ſolar action in both caſes. If the moon fell either from C or D towards the ſun, and the earth from T, their

diſtance

distance from each other would decreaſe by their falling in right lines that meet in the ſame center S. And it is obvious that the action of the ſun muſt diminiſh the force by which they tend towards each other where it would increaſe their diſtance if they fell towards the ſun, that is at A and B; but muſt increaſe that force where it would cauſe them to approach to each other, that is at C and D.

472. Let ST repreſent the gravity of the earth towards the ſun, and if *Sf* be taken upon SE in the ſame proportion to ST as the ſquare of ST is to the ſquare of SE, *Sf* will repreſent the gravity of the moon at E towards the ſun. Let *fg* parallel to ET meet ST in *g*, and the force *Sf* being reſolved into *Sg* (which acts at E in a right line parallel to TS) and *fg* which acts in the direction ET, Sir ISAAC NEWTON neglects the part ST of the force *Sg*, becauſe it is equal to the gravity of the earth towards the ſun, and acts in a parallel line; and, becauſe of the vaſt diſtance of the ſun, he conſiders *fg* as equal to ET, (the rather that as *fg* exceeds ET when E is in the part of the moon's orbit CAD, it is leſs than ET in the part DBC, and the exceſs in the former is nearly equal to the defect in the latter caſe, when the angle CTE is the ſame;) and, *Sf* being ſuppoſed to meet CD in *k*, *kf* is nearly double of *Ek*, and *Tg* nearly equal to 3TN. If TN be to N*n* as the ſquare of ST is to the ſquare of TN, the mean value of *Tg* will be more nearly equal to 3TN + 10 N*n*; and a conſtruction of the trajectory may be derived from the preceeding articles upon this ſuppoſition likewiſe; but as this conſtruction is more complex, we ſhall ſuppoſe here (with Sir ISAAC NEWTON) that *Tg* is equal to 3TN; ſo that the trajectory may be ſuppoſed to be deſcribed by theſe three forces, the gravity of the moon towards the earth, a force directed towards the earth that is to the gravity of the earth towards the ſun as ET the diſtance of the earth and moon is to ST the diſtance of the earth and ſun, and a third force *Ek* that acts in a right line parallel to TS, which is to the ſecond force as 3TN is to ET. Let G repreſent the gravity of the moon towards the earth at the diſtance TC, V the force *fg* which the ſolar action adds to this gravity when the moon is in quadrature to the ſun at C, S the periodic time of the earth a-

bout

bout the sun, L the periodic time of the moon about the earth, *l* the time in which a circle would be described about T at the distance TC by G + V the whole force at C towards T, and *l* the time in which a circle would be described at the same distance by the force G only. Then by art. 432. G the gravity of the moon towards the earth is to the gravity of the earth towards the sun in the ratio compounded of the ratio of TC to ST and of SS to *ll*: and the gravity of the earth towards the sun is to V as ST is to TC; consequently G is to V as SS is to *ll*. By the same article, *ll* is to *l l* as G + V is to G; and therefore SS is to *ll* as G + V is to V. This being premised, the trajectory that would be described by the moon, if the gravity towards the earth was varied in the same proportion as the distance, and we should abstract from the angular motion of the right line TS, may be constructed in the following manner.

473. First let the moon be supposed to set out from L in any direction LZ that is in the plane LTS; let the velocity of the motion be represented by LZ and be resolved into LY parallel to CT and YZ parallel to TS; and let LF and LG be perpendicular to TC and TA in F and G. Upon the right line TC take TB the distance from T by falling from which to F a velocity would be acquired at F equal to LY, and TA the distance by falling from which to G in the right line AG a velocity would be acquired at G equal to YZ. From the center T describe the circle BH*b* meeting FL in H, and the circle A*l*a meeting GL in I, and join TH and TI. Then in order to construct the trajectory draw any right line TP meeting the circle BH*b* in P, draw TQ so that the angle ITQ may be always to the angle HTP in the subduplicate ratio of G — 2V to G + V, and let TQ meet the circle A*l*a in Q. Then PM a right line through P parallel to TA shall always intersect QN a right line through Q parallel to BT in E a point of the trajectory: And the ark LE of this trajectory shall be described in the same time that the ark HP is described by a body revolving with an uniform motion in the circle BP*b* by a force that is to G + V as TB is to TC. This construction is deduced from art. 470. from which it follows that if Bb represent the whole force at B to-

FIG. 206.

Tb towards T and Tb meet PM in *m*, *Aa* the force at A towards T and *Ta* meet QN in *n*, the fluxion of the base TM shall be to the fluxion of the ordinate ME (or TN) in the subduplicate ratio of the trapezium BMmb to the trapezium ANna, that is, in the ratio compounded of that of PM to QN and of the subduplicate ratio of the forces at equal distances from T in the right lines TC and TS or of $\sqrt{G+V}$ to $\sqrt{G+2V}$. The fluxion of the angle HTP is to the fluxion of ITQ in the ratio compounded of that of the fluxion of TM to the fluxion of TN and of that of QN to PM. Therefore the fluxion of the angle HTP is to the fluxion of ITQ as $\sqrt{G+V}$ to $\sqrt{G-2V}$; and these angles themselves are in the same ratio by art. 24. Whence the construction is manifest. We suppose V to be less than G in this construction. If V was equal to G, the trajectory would be the same that was constructed in art. 459. when the gravity was supposed to act in right lines perpendicular to a given right line and to be as the distance from it. If V was greater than G, the construction would depend on the logarithms, or on hyperbolic areas, but it would be of no use for our present purpose to describe that case.

Fig. 108.

1474. The fluxion of TM is to the fluxion of HP, or the velocity of the point M to the velocity of P, as PM to TB; but the velocity of M is as PM; therefore the velocity of P is as TB, so that its motion is uniform and is the same with which the circle BHP is described by a force directed toward T that is to $G+V$ as TB is to TC. The motion of Q is likewise uniform and is the same with which the circle IQA is described when the centripetal force is directed towards T and is to $G-2V$ as TA to TC. And the arch LE is described in the trajectory in the same time that P with its uniform motion describes HP, or Q describes IQ. Let TK be taken upon TA in the same ratio to TN as $3V$ is to $G+V$. The gravity at E towards T is to G as ET to TC, by our supposition; the force added by the solar action at E is to V in the same ratio; and the sum of these forces is to $G+V$ in that ratio likewise. The third force Ek (described in art. 472.) is to V as 3TN to TC, and to $G+V$ as TK to TC; therefore the force Ek is to the sum of the other two forces as TK to ET; consequently the force that results from

Pl. XIII. Pag. 304.

Fig. 199.

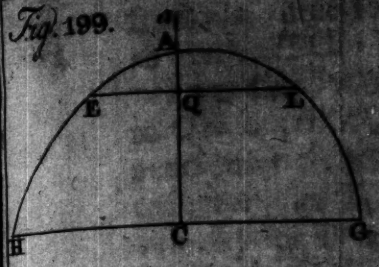


Fig. 200.

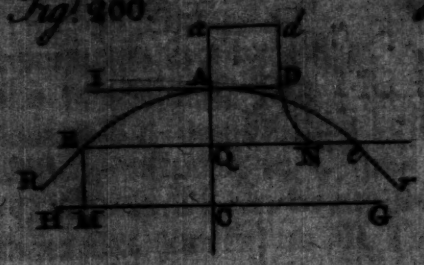


Fig. 200. N. 2.

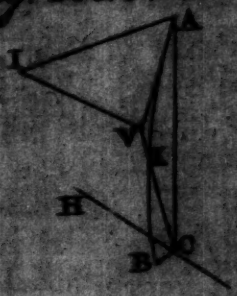


Fig. 203. N. 2.

Fig. 202.

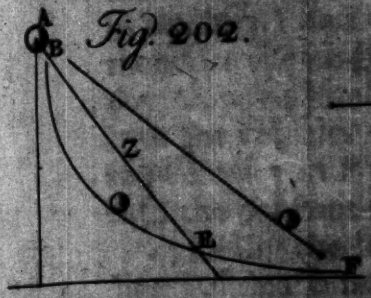


Fig. 201.

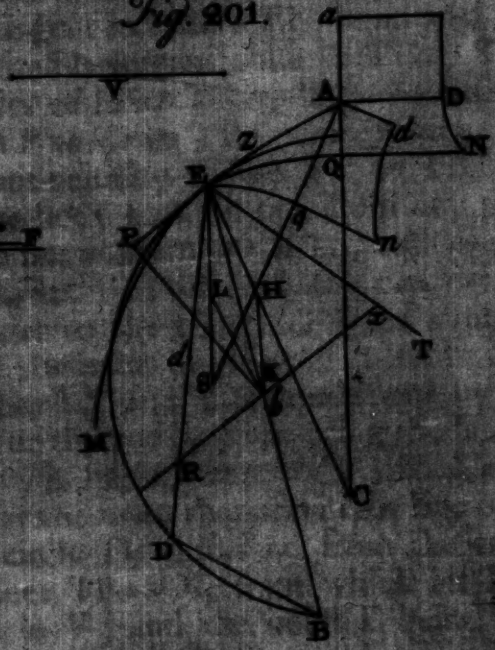


Fig. 203. N. 1.

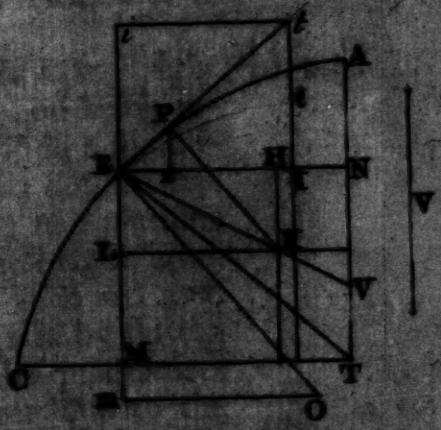


Fig. 205.

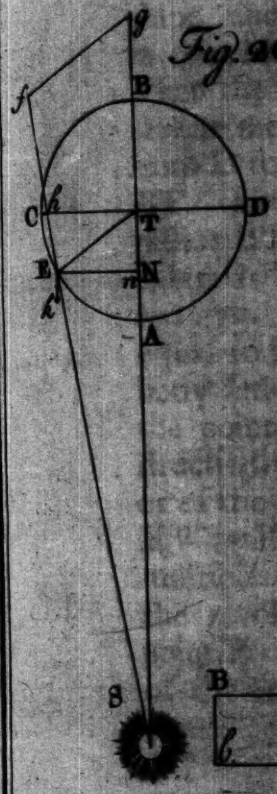


Fig. 204.

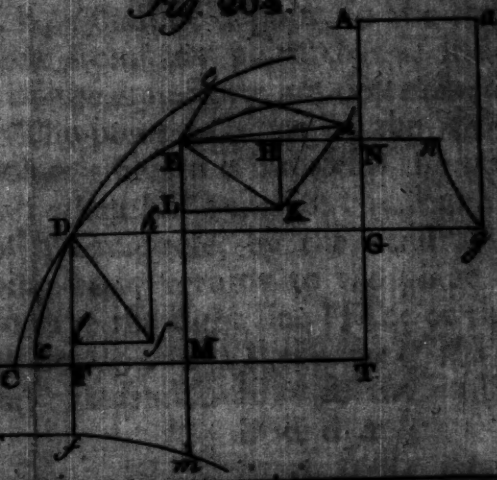
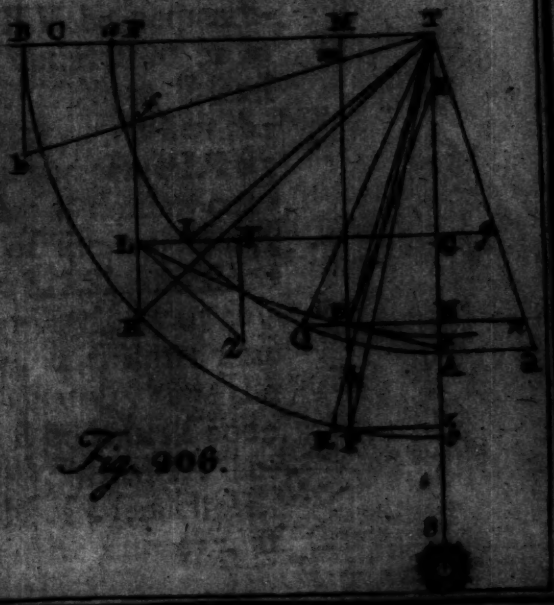
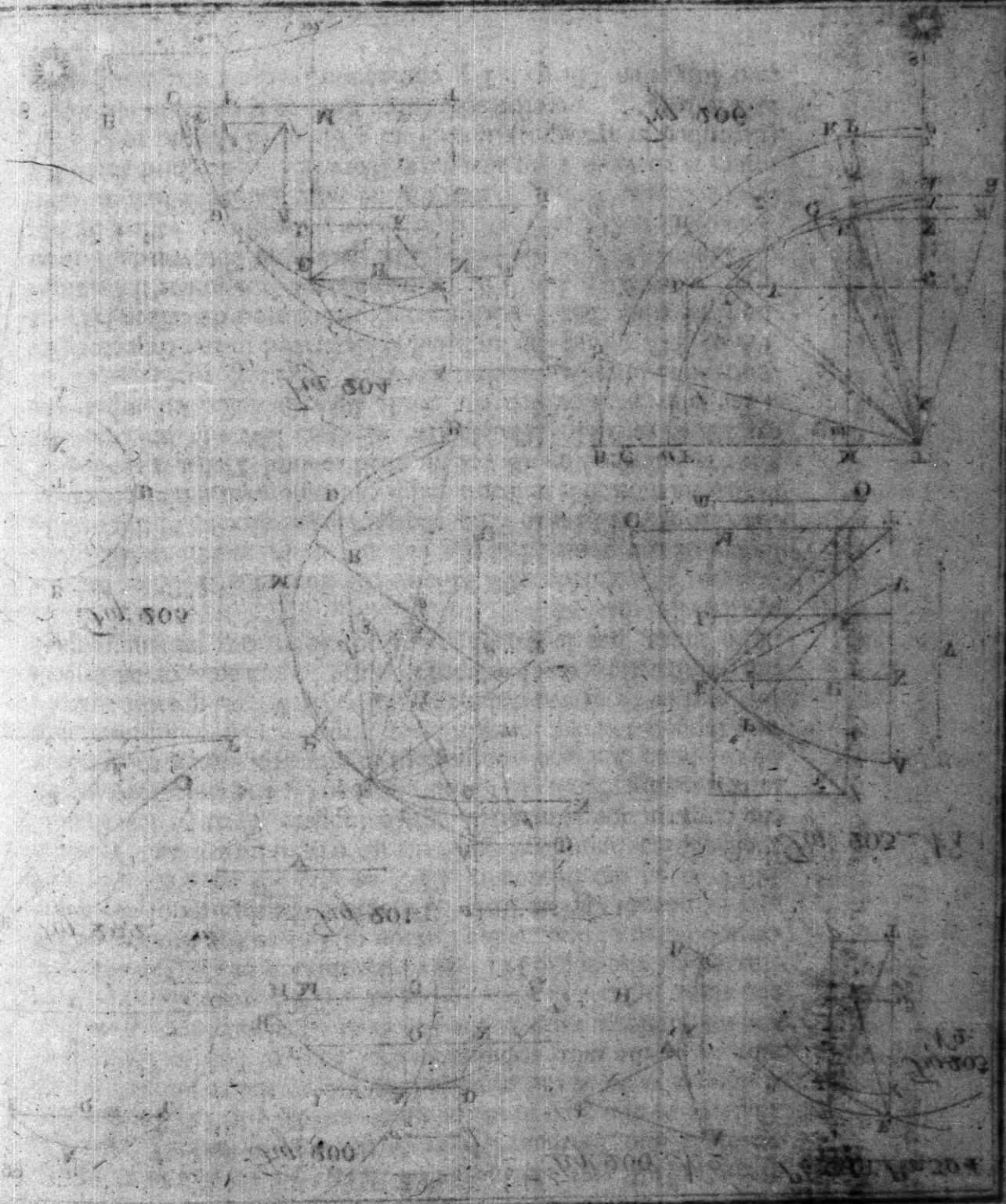


Fig. 206.





from the three forces acts in the direction EK and is to $G+V$ as EK to TC.

475. Let Pr be perpendicular to TA in r, upon rP take rR to QN as $\sqrt{G-2V}$ is to $\sqrt{G+V}$, join TR, and let Er be perpendicular to TR from E; then Er ſhall be the tangent of the trajectory at E, and the velocity at E ſhall be to the conſtant velocity of P as TR is to TB. For the fluxion of TM is to the fluxion of TN as PM (or Tr) to rR (by what was ſhewn in art. 473.) and to the fluxion of the curve LE as Ty to TR; therefore the tangent interſects EN in an angle equal to RTr, (by prop. 14.) and is perpendicular to TR. The velocity at E is to the velocity of M as TR to Tr; the velocity of M is to the velocity of P as Tr to TP; conſequently the velocity at E is to the conſtant velocity of P as TR to TB.

476. Let the body now ſet out from any point L in a right line Lſ that is not in the plane LTS, and its motion being repreſented by Lſ let it be reſolved into a motion ſZ in a direction perpendicular to the plane LTS and a motion LZ in that plane, and let the latter be reſolved as above into two motions in the directions LY and YZ repreſented by theſe right lines. Then the reſt of the conſtruction being the ſame as in art. 473. let the motion ſZ be ſuch as a body would acquire by falling from D to T by the force that acts in the line of the quadratures BT. Let a circle D*d* deſcribed from the center T with the radius TD meet TP in X, from which let XV be perpendicular to TH in V; and the point E being determined as in art. 473. if E*e* be perpendicular to the plane LTS at E and equal to XV, *e* ſhall be a point in the trajectory. When the body ſets out from any point that is not in the plane LTS; the points of the trajectory are determined by a ſimilar conſtruction. The point L is one of the *nodes* of the trajectory, or of the points wherein it interſects the plane LTS; and ſince XV vaniſhes when HP becomes equal to a ſemicircle or to any multiple of a ſemicircle, by the conſtruction, it follows that the revolving body returns to the plane LTS every time the point P comes to the right line HT*h*, or that E becomes a node of the trajectory, and TE *the line of the nodes*, when HP becomes equal to a ſemicircle or to any multiple of a ſemicircle.

FIG. 207.

FIG. 208.

477. If a rectangle DG be described so as to to touch the circle Aa in the opposite points A and a, and the circle Bb in the opposite points B and b, it appears from the construction that the curve described by the point E, (which is the projection of the trajectory on the plane LTS) will touch the four sides of this rectangle in every revolution. The points in which it touches the sides Gg, Dd that are perpendicular to Bb, move backwards, and the points in which it touches the sides dG and gD that are perpendicular to Aa, move forwards, till they come to the angles G and D. The angles HTP, ITQ and PTQ increase uniformly; and when PTQ becomes equal to a right one if TP falls upon TA, the point E will pass through T, and the revolving body will pass above T at a distance that is to TD as the sine of the angle ATH is to the radius. Thus if H coincide with B, or the body set out from the plane LTS at the quadrature B, and the excess of $\sqrt{G+V}$ above $\sqrt{G-2V}$ be to $\sqrt{G+V}$ as unit is to any integer number m, the angle PTQ will be to BTP in the same ratio. PTQ will become a right angle after TP has described as many right ones as there are units in m, the point E will then pass through T, and the perpendicular Ee will be equal to TD. The points in which the curve described by E touch Gg and Dd (which are in this case the *nodes* of the trajectory, or the points in which it intersects the plane LTS) in this time move with a retrograde motion from B to G and b to D. After this the motion of E becomes retrograde, the motion of the points where the curve LE touches Gg and Dd (or of the nodes of the trajectory) becomes direct; but the motion of the points where it touches dG and gD is then retrograde, and has always the same direction with the motion of E. And when TP has described as many right angles as there are units in $4m$, the revolving body returns to B with its first velocity and direction. The inclination of the plane of the trajectory first increases, till it become perpendicular to the plane LTS when TP has described as many right angles as there are units in m; thereafter it decreases and returns to its first magnitude in an equal time. The revolutions in this trajectory from any quadrature to the subsequent quadrature on the same side of T are compleated in times equal to each other and to the

Fig. 208.

the time in which the point Q revolves in the circle Aa by a force that is to $G - 2V$ as TA is to TC. The revolutions from any conjunction to the subsequent conjunction are compleated in the same time that P revolves in the circle Bb by a force that is to $G + V$ as TB is to TC; and these times are to each other in the subduplicate ratio of $G + V$ to $G - 2V$. What has been observed in this article holds only upon the supposition that the right line TS has no motion about the points T or S, and that the force towards T is as the distance; for when TS revolves about either of those points, as in the case of the secondary planets, the nodes move backwards in every revolution, the inclination of the plane increases and decreases within narrower limits, and the other effects of the forces *fg* and Tg are different.

478. It appears from the construction likewise that when V is much less than G, H coincides with B, and TA is nearly equal to TB, the distance TE is least not far from the octants before the syzygies, when TE bisects the angle BTA, and is greatest after the other octants when TE bisects ATb. After some revolutions the figure of the trajectory differs little from an ellipse that has its transverse axis in the latter and the conjugate axis in the former position; for if the angle QTP was invariable, TA equal to TB, and QN parallel to BT intersected PM parallel to AT always in E, (as in this construction,) the curve described by E would be an ellipse whose transverse axis would bisect the angles ATb and aTB. The tides produced in the ocean by the inequality of the gravitation of the water towards the moon rise to the greatest heights about the same octants; of this see the latter part of *Coral.* 20. *Prop.* 66. *Lib.* I. *Princip.*

479. Suppose that the body sets out from the node and quadrature B in a right line perpendicular to TB, and the rest remaining as in art. 476. it is manifest from the construction that the body will return to this node again, when the ray TP after compleating a revolution about T returns to the situation TB; in which time the ray TQ describes an angle that is to four right ones in the subduplicate ratio of $G - 2V$ to $G + V$. Therefore if the angle BTq be constituted on that side of TB which FIG. 208.

which is towards the oppoſition *a* in the ſame ratio to four right ones as the exceſs of $\sqrt{G+V}$ above $\sqrt{G-2V}$ is to $\sqrt{G+V}$, and Tq meet the circle Aa in q, then qe be drawn parallel to TB and meet Be parallel to AT in e, the point e ſhall be the node of the trajectory after this revolution of P, and the angle BTe will ſhew how far the line of the nodes has moved backwards during this revolution. Let the motion at B be reſolved into fZ perpendicular to the plane BTS (which repreſents the ecliptick) and Bz parallel to TA, and if the motion at B be given, or Bf (which repreſents this motion) be invariable, the motion BZ will vary in the ſame ratio as the coſine of the angle fBZ which is the inclination of the plane TBf to the ecliptic. But the right line TA by falling along which this motion would be acquired varies in the ſame proportion as this motion, Be is equal to the ſine of the angle BTq the radius being Tq or TA, and the angle BTq is given when the ratio of G to V is given; conſequently Be varies in the ſame ratio as the coſine of the inclination fBZ; that is the tangent of the angle BTe is as the coſine of the inclination; and when V is much leſs than G, the angle BTe varies nearly in the ſame proportion. Therefore when the velocity at B with the ratio of G to V are given, the angle TBf is right and V is much leſs than G, the retrograde motion of the node during this revolution is nearly as the coſine of the inclination of the plane.

480. In order to eſtimate from this conſtruction what ought to be the motion of the nodes of the moon nearly, we muſt ſuppoſe this trajectory to be as near to a circle as poſſible, and the body to revolve in it in the ſame time that the moon revolves about the earth. Firſt let the line of the nodes be in quadrature to the ſun about the middle of the month, and the inclination of the plane be ſuppoſed incomparably ſmall; then TA being ſuppoſed equal to TB (that the orbit may be nearly circular) and TB equal to TC, let the time in which the body E revolves from the quadrature in TB to the ſubſequent quadrature in the right line Tb be equal to the time in which the moon revolves betwixt theſe quadratures. Then ſince XV vaniſhes when HP becomes equal to a ſemicircle, it follows that E revolves from the node in TB to the ſubſequent node *n* in the time

FIG. 209

P makes

P makes half a revolution about T in the circle B*b*; but E revolves from the quadrature in TB to the ſubſequent quadrature in T*b* in the time Q makes half a revolution about T in the circle A*a*; and ſince the direct motion of the node while E proceeds from the right line T*n* to the quadrature in the right line T*b* is ſo exceeding ſmall that it may be neglected (as will appear better afterwards,) it follows that the mean motion of the node is to the mean motion of the moon in this month as the difference of the times in which Q and P revolve about T is to the time in which Q revolves about T, that is as the difference of $\sqrt{G+V}$ and $\sqrt{G-2V}$ is to $\sqrt{G+V}$. Let S and L (as in art. 473.) repreſent the periodic times of the earth about the ſun and of the moon about the earth; then ſince *l* is equal to the time in which P revolves about T in the circle B*b* by the force $G+V$, and L is equal to the time in which Q revolves about T, *ll* will be to LL as $G-2V$ to $G+V$. But by art. 473. SS is to *ll* as $G+V$ is to V; conſequently SS is to LL as $G-2V$ is to V; and $G+V$ is to $G-2V$ as $SS+3LL$ is to SS. Therefore when the inclination of the orbit is incomparably ſmall, and the orbit nearly circular, the mean motion of the node is to the mean motion of the moon in the month when the line of the nodes is in quadrature to the ſun nearly as the exceſs of $\sqrt{SS+3LL}$ above S is to $\sqrt{SS+3LL}$. And if there are 2139 revolutions of the moon to the ſtars in 160 ſidereal years (and conſequently SS to LL in the duplicate ratio of theſe numbers), this proportion is that of 1 to 120,647. By the principles laid down in the excellent treatiſe concerning *the laws of the moon's motion according to gravity*, this proportion is about that of 1 to 120,639. If we would compare it with the ratio that reſults from Sir ISAAC NEWTON's method, their difference will appear very ſmall if we may adapt his method to our preſent ſuppoſition in the following manner. It appears from what was ſhewn above (art. 432.) that when a force acts upon a body in a right line perpendicular to the direction of its motion, the deflexion of its courſe from a right line (or the angular velocity of the right line that is the direction of its motion and is always the tangent of the trajectory at the body) is as that force when the velocity is given, and in

general

general as the force directly and velocity inversely. Hence (according to his method) the motion of the node when the moon is at the conjunction is to the inflexion of the course of the moon from a right line there, as the force that produces the motion of the node at the conjunction to the force that acts upon the moon there, that is as $3V$ to $G - 2V$; and this inflexion of the orbit of the moon is to the inflexion of Q in describing the circle Aa (which Q describes by the same force $G - 2V$) as the velocity of Q is to the velocity of the moon at the conjunction, or to the velocity of P in the circle Bb, that is as $\sqrt{G-2V}$ is to $\sqrt{G+V}$; so that, according to this method, the motion of the node at the conjunction A is to the motion of Q, or the mean motion of the moon, in the ratio compounded of the ratio of $3V$ to $G-2V$, and of the subduplicate ratio of $G - 2V$ to $G + V$, that is in the ratio of $3LL$ to $8\sqrt{55+3LL}$. According to which proportion the mean motion of the node in the month when the line of the nodes is in quadrature to the sun and the plane of the orbit is supposed almost coincident with the ecliptic, is to the mean motion of the moon as 1 to 120,643. And these three proportions agree nearly, the last of which is almost a mean betwixt the other two.

481. The mean motion of the node being determined when the planes are almost coincident, if this motion be diminished in the ratio of the cosine of the inclination of the plane of the moon's orbit to the radius, we shall obtain the mean motion of the nodes of the moon in this month nearly, by art. 479. If we diminish the motion of the node that was deduced at the end of the last article in the ratio of the cosine of $4^{\circ}59'35''$. (the inclination at the syzygies in this month) to the radius, the mean motion of the node in this month (according to this method) shall be to the mean motion of the moon as 1 to 121,1023, and the mean hourly motion of the node $16' 19'' \frac{2}{15}$. If we diminish the motion of the node deduced from our construction in the same ratio, the mean motion of the node will be to the mean motion of the moon as 1 to 121,10648, and the mean hourly motion of the node will be about $16' 19'' \frac{1}{5}$. The same hourly motion of the node by the principles that are pro-

proposed in the treatise of *the laws of the moon's motion* according to gravity is about 16″ 19‴ $\frac{3}{10}$. How much the motion of the node amounts to in any of the trajectories constructed in art. 476. in a revolution or any part of a revolution will appear from what follows.

482. Supposing first H to coincide with B as in art. 479. let Pl perpendicular to TP meet TB in *l*, and *ln* parallel to AT meet the tangent at E (which is perpendicular to TR by art. 473.) in *n*, join T*n*, and it shall be the line of the nodes when the revolving body comes to *e* after describing the arch of the trajectory B*e*. For supposing the tangent of the trajectory at *e* to meet *t*E (the tangent of the curve described by E) in any point *n* and *nb* parallel to BT to meet PM in *b*; then because E*e* is always equal to XV, the fluxion of E*e* shall be to the fluxion of TV as TV is to XV; the fluxion of TV is to the fluxion of TM as XV is to PM; therefore the fluxion of E*e* is to the fluxion of TM as TV is to PM; consequently E*e* is to *bn* as TV to PM; and *bn* is to PM as E*e* (or XV) to TV, or as PM to TM, or M*l* to PM; so that *bn* is equal to M*l*, and *ln* parallel to TA; whence the construction is manifest. When Pl becomes parallel to TB, the line of the nodes becomes parallel to the tangent at E, or perpendicular to TR. The velocity at *e* in the trajectory is to the constant velocity of P in the subduplicate ratio of the sum of the squares of TR and TV to the square of TB. The ratio of TA to TB is compounded of the ratio of the velocity in the trajectory at B to the velocity of P, the ratio of the cosine of the inclination of the plane to the radius, and that of $\sqrt{G+V}$ to $\sqrt{G-2V}$. FIG. 210.

483. Let *m*N be to QN as $\sqrt{G-2V}$ to $\sqrt{G+V}$, and *no* parallel to TP meet Pl perpendicular to TP in *o*, then the angle PT*o* shall be equal to BT*n* which shews how far the line of the nodes moves from the quadrature while the arch B*e* is described in the trajectory. For let *nb* meet AT in *u*, join *mu*, and let Pl meet TA in *p*; then since the tangent E*n* is perpendicular to TR; PM is to R*r* (or *m*N) as *nb* (or M*l*) is to E*b*, or *u*N; consequently *m*N is to *u*N as R*r* to T*r*, and *um* is parallel to TP. Therefore *ln*, or T*u*, is to P*o* as T*p* is to P*p*, or as T*l* to TP, and the angle PT*o* is equal to *l*T*n*. Hence if the

arch bk be to a ſemicircle as the exceſs of $\sqrt{G+V}$ above $\sqrt{G-2V}$ is to $\sqrt{G-2V}$, Tg be taken upon Tk in the ſame ratio to TA as $\sqrt{G-2V}$ to $\sqrt{G+V}$, and gf parallel to Tb meet bf perpendicular to Tb in f, the angle bTf ſhall be equal to the motion of the nodes in this trajectory in the time the body revolves from the quadrature B to the ſubſequent quadrature. When TA is equal to TB, and the planes are almoſt coincident; this angle bTf is to two right ones in the ratio compounded of the ratio of the exceſs of $\sqrt{G+V}$ above $\sqrt{G-2V}$ to $\sqrt{G+V}$, and of that of the coſine of bTk to the radius nearly.

FIG. 211. 484. Suppoſe now that the line of the nodes is in any ſituation TL at the beginning of a revolution as in art. 476. and in order to determine the motion of the node during the revolution, take the arch Iq backwards from I in the ſame ratio to a whole circle as the exceſs of $\sqrt{G+V}$ above $\sqrt{G-2V}$ is to $\sqrt{G+V}$, and let ql parallel to BT meet LF in l, join Tl, and the trajectory ſhall interſect the plane LTS in l after a compleat revolution of the point P. For when P returns to H, XV vaniſhes, and the arch deſcribed by Q is to a whole circle as $\sqrt{G-2V}$ to $\sqrt{G+V}$, by the conſtruction. Therefore when P returns to H, Q comes to q, and E to the point l in the plane LTS. The angle $LT\mathit{l}$ ſhews how far the nodes have moved backwards in this time, and is to the angle ITq as the rectangle LGI to the ſquare of $T\mathit{l}$ nearly when V is much leſs than G. For let $l\varkappa$ be perpendicular to $T\mathit{l}$ in $\varkappa$, and Io parallel to AT meet ql in o, then $L\varkappa$ ſhall be to LI (or Io) as LG to TL, and Io to Iq as IG to TI, nearly, conſequently $L\varkappa$ is to Iq as the rectangle LGI to the rectangle contained by TL and TI, and the angle $LT\mathit{l}$ to ITq, as the rectangle LGI to the ſquare of TL, nearly. Hence when TA is equal to TB, V is much leſs than G, and L is taken upon the circle $B\mathit{l}$, if the ratio of V to G be given, the angle ITq is given, and the motion of the nodes (or the angle $LT\mathit{l}$) is as the ſquare of LG the ſine of the angle LTA the diſtance of the node from the ſun; which is agreeable to what is ſhewn of the nodes of the moon by Sir ISAAC NEWTON, *cor.* 2. *prop.* 30. *lib.* 3. *princip.* Suppoſing therefore the motion of the nodes of the moon to vary in the duplicate ratio of the ſine of their diſtance from the ſun, the annual motion of the nodes may be briefly

Fig. 207.

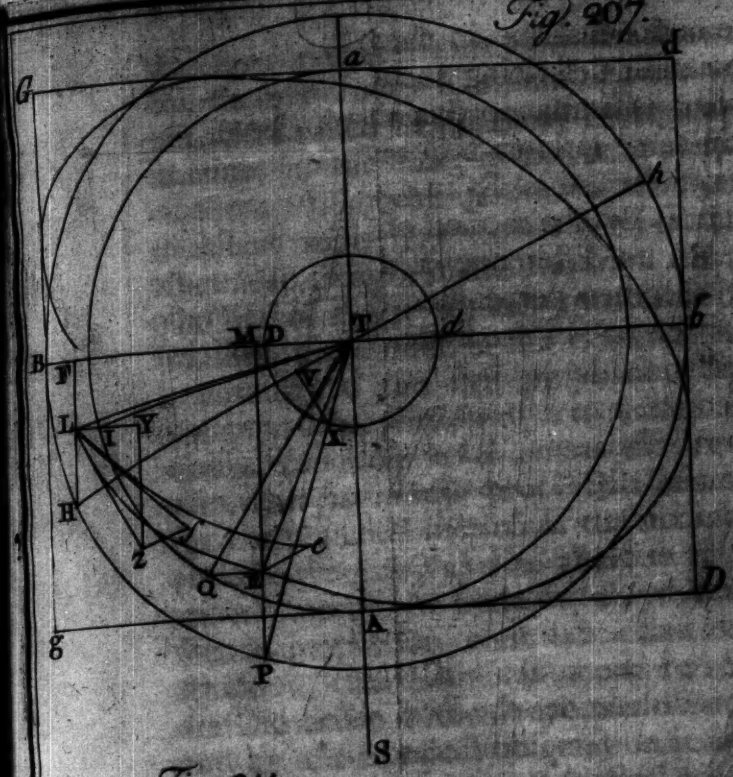


Fig. 208.

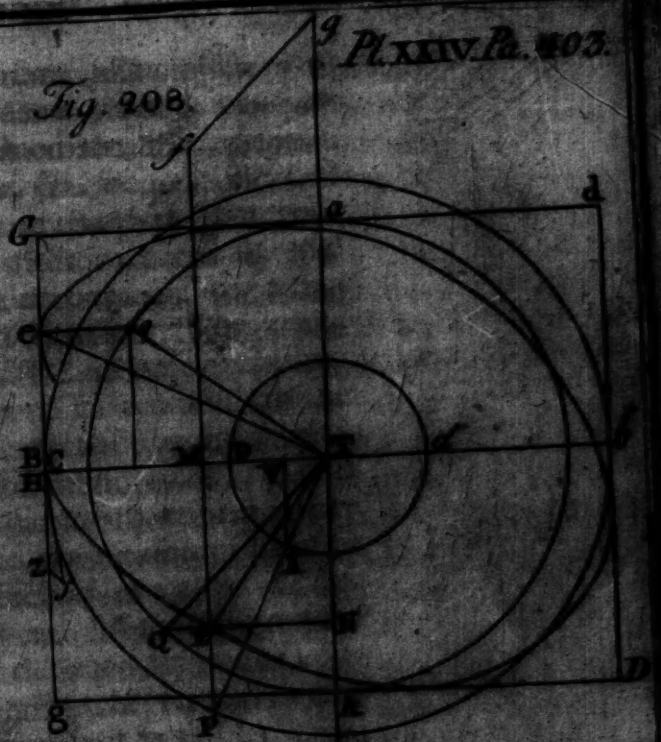


Fig. 211.

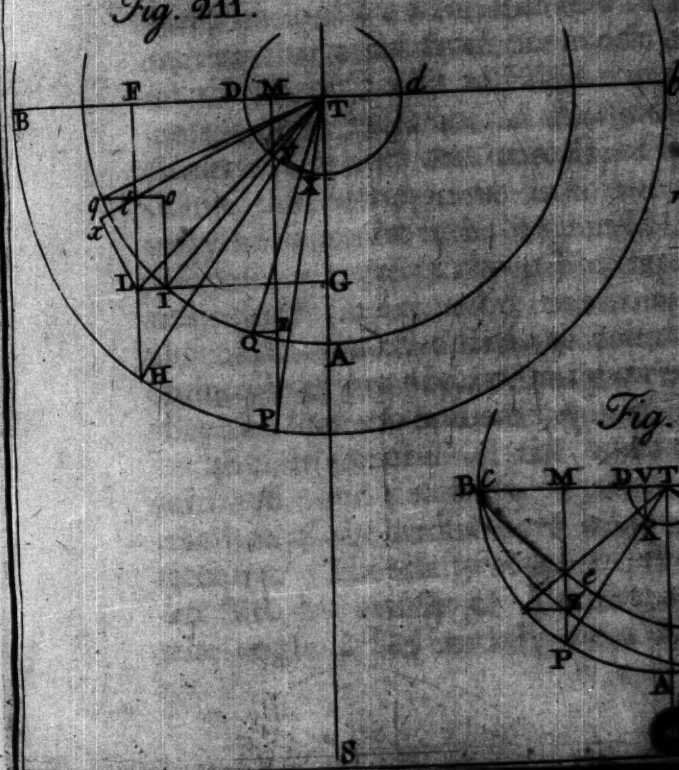


Fig. 210.

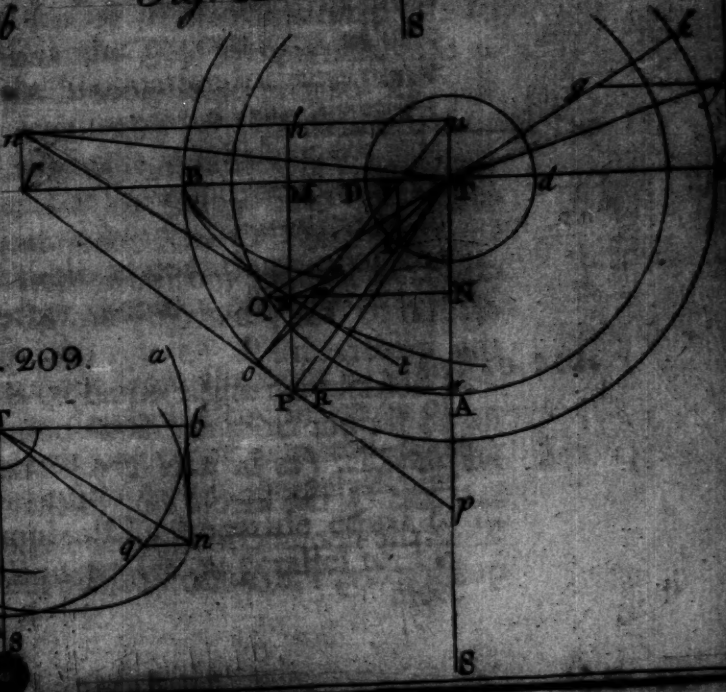
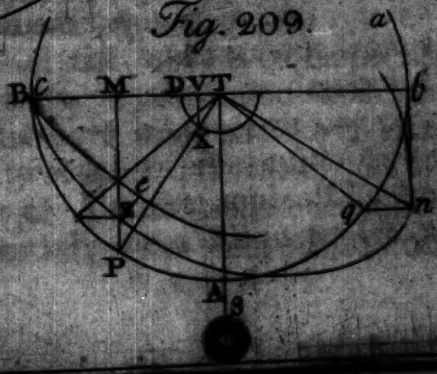
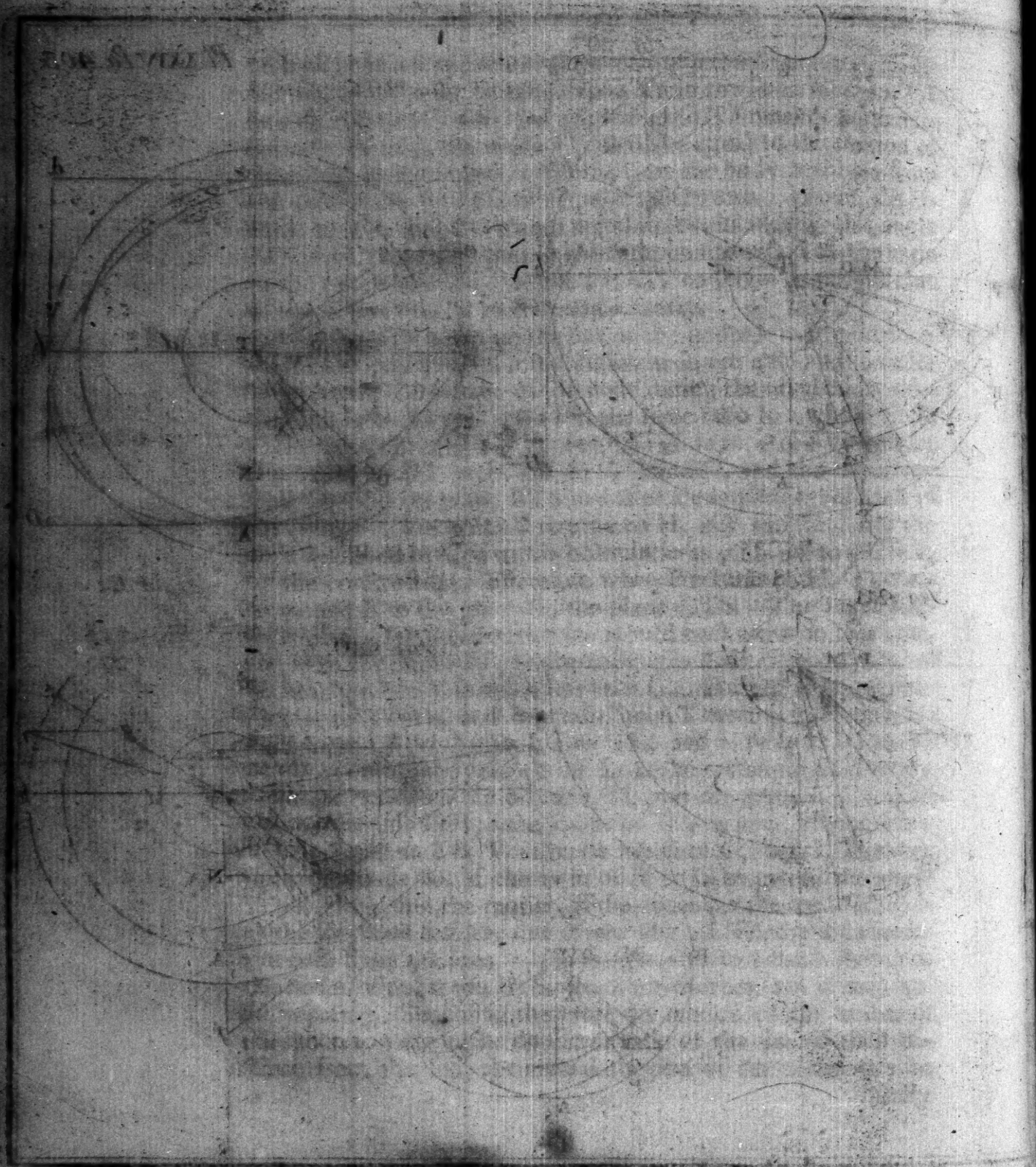


Fig. 209.





briefly computed from what was ſhewn in art. 481. by the uſeful theorem in the ſcholium of *prop.* 33. *lib.* 3. *princip.* communicated by Mr. MACHIN and ſince explained in a more general manner in the *laws of the moon's motion*, &c. *p.* 14. By that theorem, the mean motion of the ſun from the node is a geometrical mean proportional between the motion of the ſun and the mean motion of the ſun from the node in the month when the line of the nodes is in quadrature to the ſun. The mean motion of the node in this month is to the motion of the ſun, according to the firſt proportion deduced in art. 481. in the compound proportion of 1 to 121,1023, and of the mean motion of the moon to the mean motion of the ſun, or of 2139 to 160, that is in the proportion of 1 to 9,05861; conſequently the mean annual motion of the node is to the motion of the ſun (by the theorem) as 1 to 18,60413; which gives 19° 21′ 2″ for the motion of the node in a ſydereal year. According to the ſecond proportion in art. 481. that was deduced from our conſtruction, this motion is about 19° 21′; by the principles that are laid down in the above mentioned treatiſe it is 19° 21′ 7″ ½; and by aſtronomical obſervation it is 19° 21′ 21″ ½. In theſe computations we have abſtracted from the acceleration of this motion that ariſes from the exceſs of the ſolar force on that half of the moon's orbit which is towards the ſun above what it is on the other half, and from any effect the excentricity of the lunar orbit, and the increaſe of the inclination of the plane while the nodes move from the quadratures to the ſyzygies, may be ſuppoſed to have upon it. But this is ſufficient for our purpoſe, which is only to illuſtrate from the conſtruction of the trajectory (in art. 476.) the computation of this motion from its cauſe, which affords ſo remarkable a confirmation of the principle of gravity, and may on other accounts be of great uſe. See the *laws of the moon's motion*, *p.* 24.

485. In general let the body ſet out from the node L, as in art. 476. and let it be required to determine the ſituation of the line of the nodes when it comes to any point *e* in the trajectory. The tangent E*t* of the curve LE is perpendicular to TR by art. 475. Let the angle TP*p* be made equal to the angle ATH and P*p* meet TB in *p*; let *pn* parallel to TA meet

FIG. 213.

meet *Et* in *n*, join *Tn*, and the nodes shall be in the right line *Tn* when the body comes to *e* after describing the arch *Le* in the trajectory. The demonstration is similar to that in art. 482. If *m*N be taken upon NQ in the same ratio to NQ as $\sqrt{G-2V}$ is to $\sqrt{G+V}$, and *mo* be perpendicular to *Pp* in *o*, the angle HT*n* will be equal to *o*TP, and LT*n* the motion of the node will be either equal to the sum or difference of the angles *o*TP and LTH. When TP becomes perpendicular to TH, *Pp* becomes parallel to BT and the line of the nodes perpendicular to TR.

486. In art. 479. we supposed the body to set out from B in a right line perpendicular to TB. But if it set out from any
Fig. 213. point L in TB, and the right lines TB, TL with the velocity at L be given, the motion of the node will be still as the sine-compliment of the inclination of the plane. If TA be equal to TB, and the arch BH be taken from B towards the opposition *a* in the same ratio to a semicircle as the excess of $\sqrt{G+V}$ above $\sqrt{G-2V}$ is to $\sqrt{G+V}$, the line of the nodes will describe an angle equal to BTH in the time P describes the semicircle H*b*; because P and Q will come at the same time to the point *b*. This is the motion of the node that was determined in the first part of art. 180. Supposing still the node L to be in the line of the quadratures TB, let the angle BTH be to a right angle as the difference of $\sqrt{G+V}$ and $\sqrt{G-2V}$ is to $\sqrt{G-2V}$, (or in the case of the moon according to art. 180. as 1 to 119,6469) and the arch BH be taken from B towards the opposition; let the point P describe the circle B*b*, and Q the circle LA, TA being supposed equal to TL; then (by art. 476.) the right lines TP and TQ will coincide with each other at the conjunction, where the motion in the trajectory will become perpendicular to TA; and (by art. 485.) while the body moves from the quadrature L to the conjunction A, the line of the nodes will describe an angle equal to BTH. But if we suppose a body to set out from the same point L in a right line perpendicular to TL, (as in art. 479.) the point P to describe the circle LA and Q the circle B*b*, so that the distance of the body from T at the conjunction may be equal to TL its distance at the quadrature, the motion of the node while the body moves from the

the quadrature to the conjunction will be equal to the angle ATR if the angle BTA be to BT*q* as $\sqrt{G+V}$ to $\sqrt{G-2V}$, and *qn* be to R*n* in the same ratio, by art. 482. and this angle ATR is to a right one as 1 to 121,65526. If we suppose the motion of the node in a circular orbit, when the planes are almost coincident, to be a mean betwixt the motion of the node in those two cases, it will be to the mean motion in the orbit as 1 to 120,64273. This almost coincides with the proportion that was deduced for this case at the latter end of art. 480. in imitation of Sir ISAAC NEWTON's method; which in larger numbers is that of 1 to 120,64275.

487. The angle in which the plane of the lunar orbit intersects the ecliptick is perpetually varying; and this motion may be also illustrated from art. 476. In general if EK be perpendicular to the line of the nodes T*n* in K, the inclination of the plane of the trajectory to the plane BTA is an angle the tangent of which is to the radius as XV to EK. First let the line of the nodes be in quadrature to the sun, and suppose the body to set out from the quadrature in a right line perpendicular to TB, as in art. 479. make the angle ST*b* equal to the inclination at the quadrature; take TZ upon T*b* in the same ratio to TB as the velocity at B in the trajectory is to the velocity of P in the circle B*b*; let Z*d* parallel to ST and T*d* perpendicular to ST meet in *d*; upon *d*Z take *da* to *d*Z as $\sqrt{G+V}$ is to $\sqrt{G-2V}$, join Ta, and the angle STa will be nearly equal to the inclination at the syzygies. For TZ is equal to the semiaxis that is conjugate to TB in the ellipse that would have been described about T if the third force T*g* (described in art. 472.) had not acted; and TA is to *d*Z as $\sqrt{G+V}$ to $\sqrt{G-2V}$; consequently TA is equal to *da*. But at the syzygies the distance of the point E from T is nearly equal to TA, and Es to Td. Therefore the angle STa is the inclination at the syzygies nearly, and the angle aTZ is the decrement of the inclination while the body moves from the quadratures to the syzygies. Let Z*y* be perpendicular to Ta in *y*, and because Z*y* is to Za as T*d* is to Ta, the angle aTZ is nearly to Ta*d* (or STa) as Za is to TA, that is in the ratio compounded of that of za to *da* (or of the excess of $\sqrt{G+V}$ above $\sqrt{G-2V}$ to $\sqrt{G+V}$) and that of *da* to

FIG. 212.

FIG. 214. *and* 209.

Ta

Ta (or of the coſine of the inclination to the radius.) But the mean motion of the node in this revolution is to the mean motion of E in the ſame ratio, by art. 481. Therefore (by applying this to the moon) the exceſs of the inclination at the quadratures above what it is at the ſyzygies in this month is to the latter inclination as the mean motion of the node in this month is to the mean motion of the moon, or as 1 to 121,1; which is the ratio aſſigned in the treatiſe above mentioned, *p.* 26. where the intermediate inclinations are alſo determined in an elegant manner. According to this proportion the difference of the inclinations at the quadratures and ſyzygies is about 2′ 28″ $\frac{2}{7}$; and if this difference be increaſed in the ratio of the ſynodical to the periodical month, it will then agree nearly with that which reſults from Sir ISAAC NEWTON's method, *prop.* 33. *lib.* 3. *princip.* If the body be ſuppoſed to continue its motion in the trajectory, the inclination at the node e will be greater than the inclination at the node B at the beginning of the revolution in the ſame ratio as the radius is greater than the ſine of an arch that is to a circle as $\sqrt{G-2V}$ to $\sqrt{G+V}$.

FIG. 215. 488. Suppoſe O to be the projection on the plane BTA of *v* the place where the diſtance of the body that deſcribes the trajectory from the plane BTA is greateſt at the beginning of any revolution; and let the arch OQ be taken backwards from O on the circle OK deſcribed from the center T in the ſame ratio to the whole circumference as the exceſs of $\sqrt{G+V}$ above $\sqrt{G-2V}$ is to $\sqrt{G+V}$; let QE parallel to BT meet OE parallel to TA in E, and TE meet the arch OQ in V; then if TD perpendicular to TV be equal to O*o*, the angle EDV will ſhew nearly how much the inclination of the plane varies in this revolution; becauſe the angle TVD is equal to OT*o*, and TED is nearly equal to the inclination of the plane when the body after almoſt a compleat revolution is again at the greateſt diſtance from BTA. When TO is within the angle BTA, or *b*T*a*, TE is leſs than TO, the angle TVD is leſs than TED, and conſequently the inclination of the plane increaſes: but when TO is within the angle AT*b* or *a*TB, TE is greater than TO, and the inclination of the plane decreaſes; that is while the line of the nodes (which is perpendicular to TO) moves from the quadratures

dratures to the ſyzygies, the inclination of the plane increaſes; but while the line of the nodes moves from the ſyzygies to the quadratures, the inclination decreaſes.

489. Let the arch E$\varkappa$ deſcribed from the center D meet DV in $\varkappa$, and V$\varkappa$ be perpendicular to TA in G; then becauſe E$\varkappa$ is to EV as TD to DV, EV to VQ as VG to TG, and VQ to OQ as the ſquare of TG to the ſquare of TV, nearly, it follows that the angle EDV is to OTQ, in the ratio compounded of that of TD to DV, and that of the rectangle TGV to the rectangle contained by TV and DV. But the angle OTQ is to the motion of the node in that month when the line of the nodes is in quadrature to the ſun (which we ſhall call N) as DV to TV, by art. 479. conſequently the angle EDV is to N in the ratio compounded of the ratio of TD to DV, and that of the rectangle TGV to the ſquare of TV, that is in the ratio of the rectangle contained by the ſine of the inclination and the ſine of 2BTV (the double diſtance of the nodes from the ſun) to twice the ſquare of the radius; which agrees with *cor.* 3. *prop.* 33. *lib.* 3. *princip.* The decrement of the inclination in a revolution while the nodes move from the ſyzygies to the quadratures is nearly as EV; it is greateſt when V is in the octants, where EV becomes nearly equal to $\frac{1}{2}$ OQ; and EDV, the decrement in a revolution, is to the mean motion of the nodes nearly as TD to DV, or as the ſine of inclination to the radius. Becauſe the radius TO and the arch OQ are ſuppoſed to be given, the point E is always in an ellipſe HIk, whoſe tranſverſe axis is in the octants after the ſyzygies; and the whole decrement of the inclination while the nodes move from the ſyzygies to the quadratures is nearly to the decrement that would have been generated in the ſame time by the variation at the octants continued uniformly, as the area deſcribed by EV while O moves along the quadrant Kk is to the rectangle contained by $\frac{1}{2}$ OQ and Kk, or as the area included between the quadrants of the ellipſe and circle to that rectangle, that is as the radius TK to the quadrant Kk. Hence the mean hourly decrement of the inclination is to the mean hourly motion of the node as the ſine of the inclination (the radius being TK) is to the quadrant Kk; ſo that when the inclination is ſmall, the

whole

whole inclination would be generated by its mean hourly variation in the same time that a quadrant would be described by the nodes with their mean motion. Therefore the whole decrement of the inclination while the nodes move from the syzygies to the quadratures (or while a quadrant is generated by the motion of the sun from the node) is to the whole inclination as the mean motion of the node is to the mean motion of the sun from the node, or as 1 to 19,6 (by art. 484.) according to which proportion that decrement is about 16′ 10″. See the *laws of the moon's motion, &c. p.* 26.

FIG. 216. 490. If we suppose a body to descend along the quadrant BA, and the velocity at B to be equal to the velocity of Q as in art. 480. the velocity at A will be equal to that which would be acquired by falling in the chord from B to A or by falling from B to T and then from T to A, by art. 362. The velocity that would be thus acquired at A is the same that would be acquired if the motion was not accelerated from B to T, and was accelerated from T to A by the force Tg (*Fig.* 204) only. Let *Bb* perpendicular to TB be to *Aa* perpendicular to TA as $\sqrt{G-2V}$ is to 3V, and the velocity acquired at A will be to the velocity at B in the subduplicate ratio of the sum of the triangles TB*b*, TA*a* to the triangle TB*b*, or of the sum of B*b* and A*a* to B*b*, that is as $\sqrt{G+V}$ to $\sqrt{G-2V}$ or as $\sqrt{SS+\frac{9}{4}LL}$ to S. In like manner the velocity at any point E is determined. And this will be found to agree nearly with what is shewn concerning the acceleration of the area described about the earth in a circular orbit, *prop.* 26. *lib.* 3. *princip.* where this acceleration is computed by the increment of the velocity in such an orbit; and it coincides with what is shewn *p.* 29 of the *laws of the moon's motion*. The same increase of the velocity may be deduced from what has been shewn of the trajectory Be when its
FIG. 209. plane coincides with the plane BTS, and it is supposed nearly circular; for the velocity at B is equal to the velocity of Q; when TA is equal to TB the velocity at the syzygies is nearly equal to the velocity of P, by art. 475. and these velocities are to each other as $\sqrt{G-2V}$ to $\sqrt{G+V}$. Some other corollaries relating to this theory might be deduced from the preceeding articles.

491. If

491. If a fluid be ſuppoſed to gravitate towards two points C and S with equal forces that are the ſame at all diſtances, the figure of the fluid will be an oblong ſpheroid, and theſe two points will be the *foci* of the generating ellipſe. For ſuppoſing AE*a* to be any ſection of the fluid through C and S, let EM and EN be taken upon EC and ES repreſenting the equal forces towards the points C and S reſpectively; let MP and NQ be perpendicular to the tangent at E in P and Q, and EP will be equal to EQ becauſe of the equilibration of the fluid. The fluxion of CE is to the fluxion of the curve AE as EP to EM, and the fluxion of SE to the fluxion of AE as EQ to EN, by prop. 17. Therefore CE and SE flow with equal motions, and (becauſe SE decreaſes while CE increaſes) the ſum of CE and SE is invariable; conſequently AE*a* is an ellipſe that has its *foci* in C and S. The gravity at E is to the gravity at A as the ſine of the angle CEP is to the radius. When the forces directed towards C and S are invariable but unequal, let D be any point of the ſurface AE*a* that terminates the figure, and it will appear in the ſame manner that the difference of CE and CD will be to the difference of SD and SE as the force towards C is to the force towards S. The figures by which rays of light iſſuing from a given point point S may be refracted ſo as to have their *focus* afterwards in C are of this kind. The figure AE*a* is in ſome caſes a conic ſection, and when it paſſes through the point C or S it is a portion of an epicycloid that is deſcribed when a circle revolves on an equal circle. If equal and invariable forces are directed towards any number of given centers, the aggregate of the right lines drawn from any point in the ſurface that terminates the fluid to thoſe centers is invariable. When the forces towards the centers are inverſely as the ſquares of the diſtances, the aggregate of right lines that are inverſely as the diſtances of any point of the ſurface from thoſe centers is invariable. And the figures have an analogous property when theſe forces at equal diſtances are not equal but in a given ratio. But it would be of little uſe in philoſophy to inſiſt on this ſubject. FIG. 217.

492. Suppoſe that a fluid, which gravitates towards the point C, revolves about the axis AB; let ER be perpendicular FIG. 218.

to AB in R from any point E in AEB the ſurface of the fluid, and let CD be the ordinate at C; then it will appear (as in the laſt article) that the fluxion of CE ſhall be to the fluxion of ER, as the centrifugal force is to the gravity at E, or (becauſe the centrifugal force at E is to the centrifugal force at D as ER is to CD) in the compound ratio of the centrifugal force at D to the gravity at E and of ER to CD. From which it follows that if the gravity towards C be the ſame at all diſtances, and a right line L be to CD as the gravity is to the centrifugal force at D, the ſquare of ER will be equal to the rectangle contained by CE — CA and $\frac{1}{2}$ L, which agrees with what has been ſhewn by Mr. HUYGENS. If the gravity be inverſely as the ſquare of the diſtance from C; let G repreſent the gravity and V the centrifugal force at D, and let the ſquare of the right line L be to the ſquare of CD as G + $\frac{1}{2}$ V is to $\frac{1}{2}$ V, then the ſolid contained by CE and the ſquare of ER ſhall be equal to the ſolid contained by CE — CA and the ſquare of L, and CD will be to CA as G + $\frac{1}{2}$ V is to G. But the figure of the earth or planets is not to be diſcovered by ſuppoſitions of this kind; for as the gravity of a body reſults from the gravity of all its parts, ſo the gravitation towards any body reſults from the gravitation towards the particles of which it conſiſts, as is ſhewn by Sir ISAAC NEWTON; and when the figure of the fluid varies from a ſphere, this gravitation is not directed towards a fixed point. Of this we ſhall treat in the next chapter, where we ſhall ſhew that if the earth was of an uniform denſity, its figure would be an oblate ſpheroid accurately.

493. We have inſiſted at ſo great length on what relates to the curvature of lines, and to the application of this theory to philoſophical problems, becauſe in this conſiſts one of the greateſt advantages of the modern geometry. There are many other problems that depend on this theory, but we ſhall conclude this ſubject with obſerving, that as when a right line interſects an arch of a curve in two points, if by varying the poſition of that line the two interſections unite in one point, it then becomes the tangent of the arch; ſo when a circle touches a curve in one point and interſects it in another, if by varying the center this interſection join the point of contact, the circle

then

then has the closest contact with the arch and becomes the circle of curvature; but it still continues to intersect the curve at the same point where it touches it (that is where the same right line is their common tangent, according to the definition in art. 181.) unless another intersection join that point at the same time. Let ME*m* be any arch that is continued from E on both sides of the ray of curvature CE, ACB a right line perpendicular to CE; let the right line CM revolve about C and meet the curve in M, and PM be perpendicular to AB in P: then if AP be supposed to flow uniformly, the first and second fluxions of CM vanish when M comes to E; and, when the third fluxion of CM, or of PM, does not vanish at the same time, the circle of curvature intersects the curve at E. In general when M comes to E, and the number of the fluxions of CM of successive orders (including the first fluxion) which then vanish is even, the circle of curvature at E intersects the arch ME*m*; but if this be an odd number, there is no such intersection at E. For let PN be taken upon PM always equal to CM. Then the curve NE*n* described by N will pass through E; and it is manifest, that the circle of curvature intersects the arch ME*m* only when NE*n* intersects its tangent at E, that is when NE*n* has a point of contrary flexure at E. Therefore (by art 266.) the circle of curvature at E intersects the arch ME*m* only when the number of those fluxions of CM that vanish at the term of the time when M comes to E, is an even number. This theorem may serve to illustrate a subject that was disputed some time ago by two celebrated authors. One of them imagined that two points of contact, or four intersections, of the curve and circle of curvature must join each other to form an osculation. But Mr. James Bernovilli insisted, on just grounds, that the coalition of one point of contact and one intersection, or of three intersections, was sufficient. In which case (and in general when an odd number of intersections only join each other) the point where they coincide continues to be an intersection of the curve and circle of curvature, as well as a point of their mutual contact and osculation. Fig. 219.

494. In the preceeding chapters we deduced the principal propositions of the method of fluxions, and those upon which its

its application to the solution of problems depends, from the plain axioms concerning motion in art. 15. Having proposed this definition (art. 20.) as a method of deducing the relations of quantities, by comparing the motions which are conceived to generate them, it seem'd natural to establish this method on the most evident principles that relate to motion; and if it be no less known to us than extension, it would seem, that as this doctrine is far more general and comprehensive than the common geometry, so it cannot be said to be inferior to it in accuracy or evidence. Nor can it seem improper, to deduce the properties of figures from the same principles which serve for describing their genesis. Thus, the definition of a fluxion (art. 11.) seem'd naturally to lead us into the method of treating this doctrine which we have followed hitherto; and its connexion with the manner in which the genesis of figures is most commonly described in geometry, and with the most useful theories concerning motion in philosophy, (of which we have had some examples in this chapter,) induced us the rather to pursue it. But we have insisted on it at so great length chiefly, because a full account of the manner in which the principal propositions of the method of fluxions are demonstrated by it, may be of use for removing several objections that have been lately urged against this doctrine; which has been represented, as depending on nice and intricate notions; while it has been insinuated, that they who have treated of it have been earnest rather to go on fast and far, than solicitous to set outwardly, and see their way distinctly. But we now proceed to the more concise methods by which the fluxions of quantities are usually determined.

The End of the first Volume.

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Fig. 213.

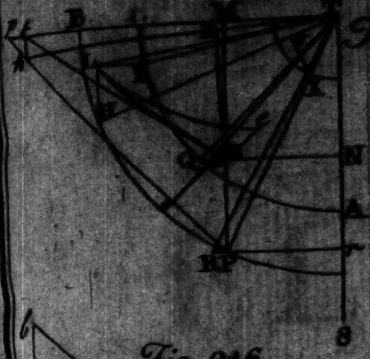


Fig. 214.

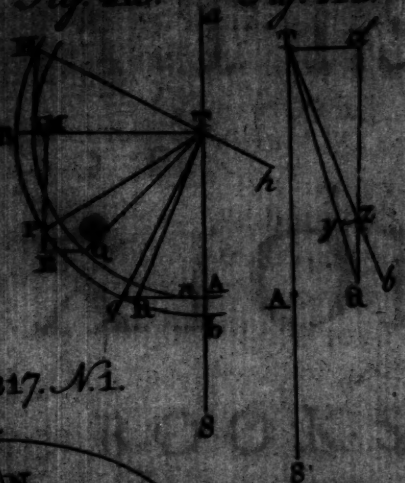


Fig. 215. Pl. XIV. Pa. 4

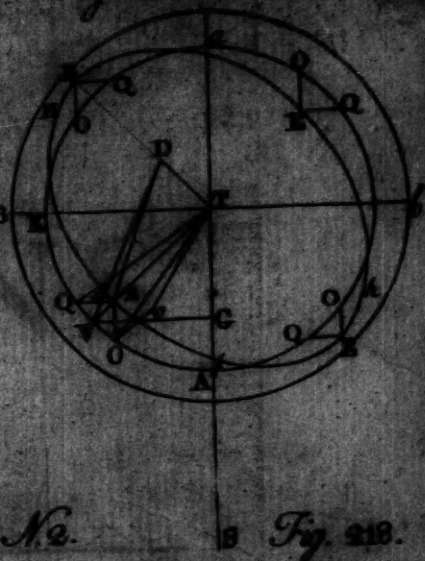


Fig. 216.

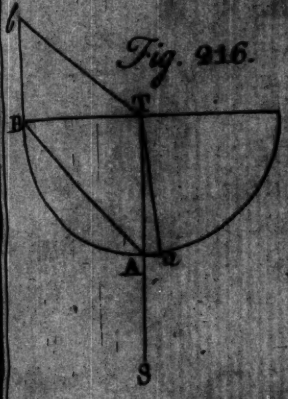


Fig. 217. N. 1.

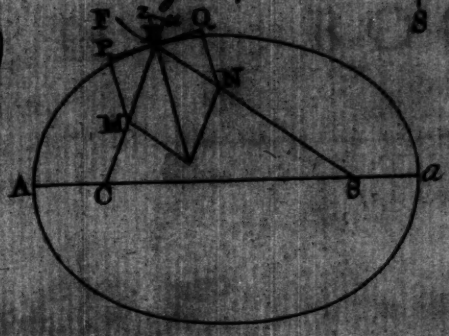


Fig. 217. N. 2.

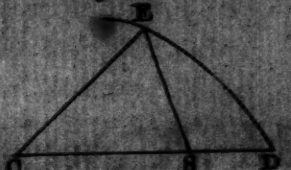
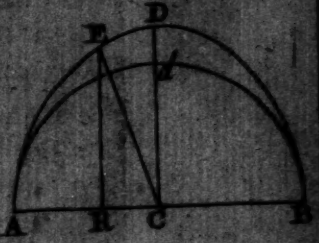


Fig. 218.



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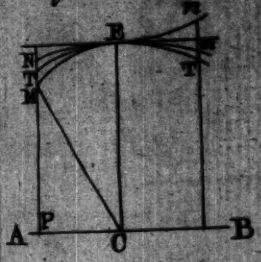

Fig. 220.

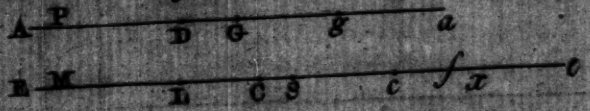

Fig. 225.

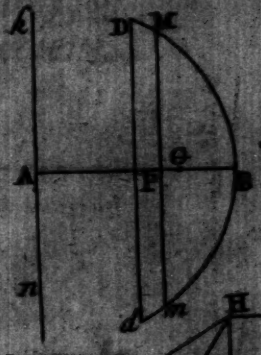

Fig. 221.

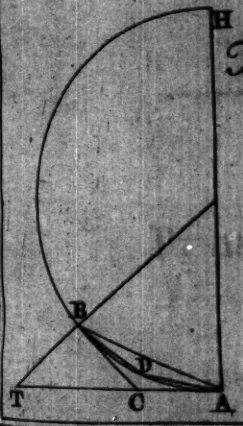

Fig. 222.

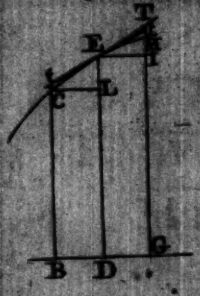

Fig. 223.

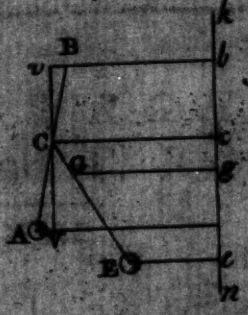

Fig. 224.